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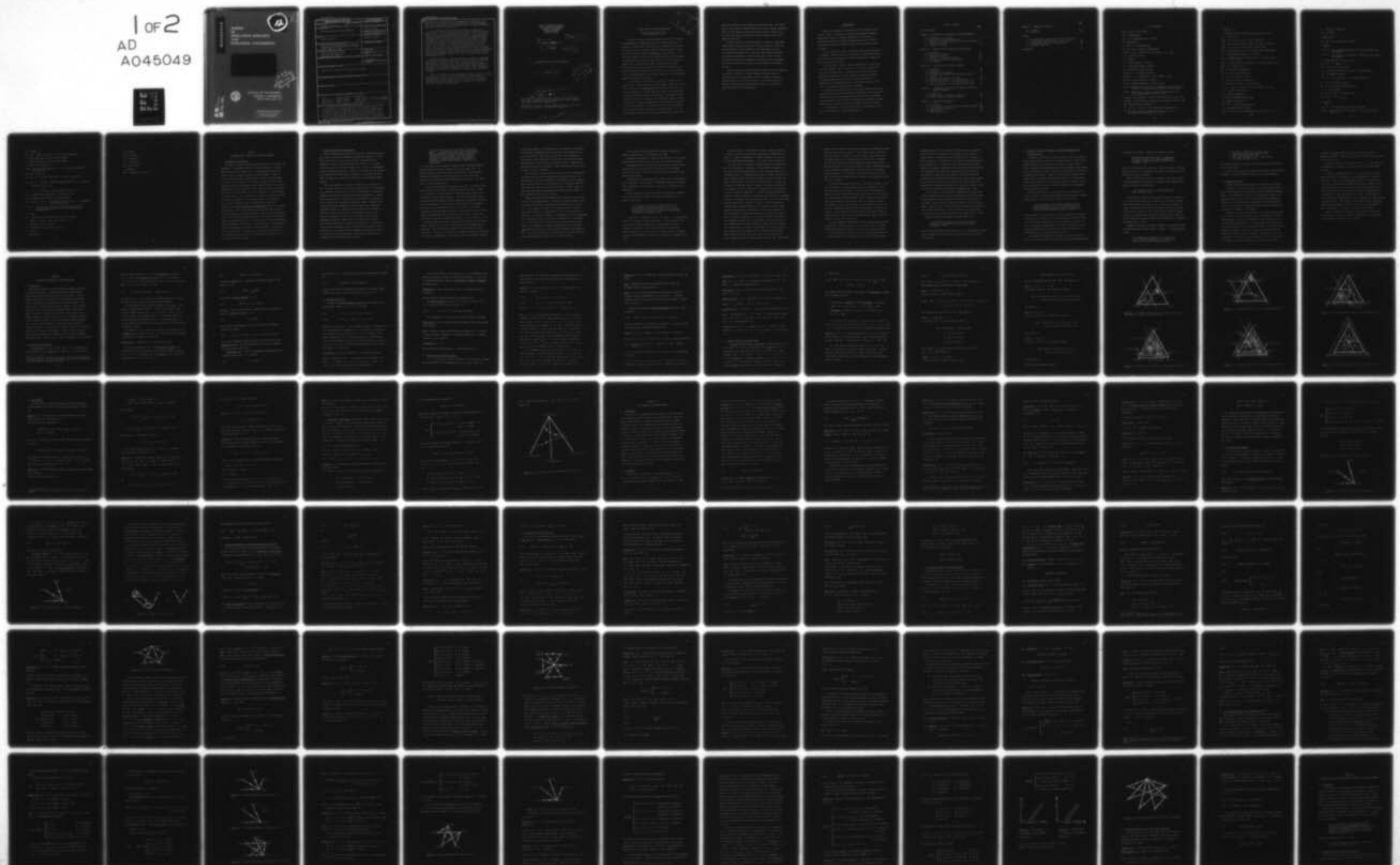

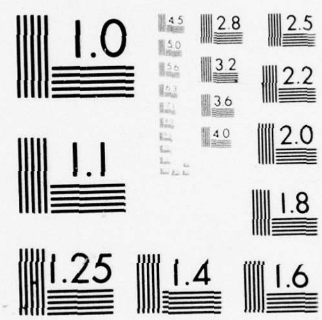
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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER Technical Report No. 342	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) ON GAME THEORY AND COALITION FORMATION		5. TYPE OF REPORT & PERIOD COVERED Technical Report 6. PERFORMING ORG. REPORT NUMBER Technical Report No. 342
7. AUTHOR(s) Shenoy, Prakash P.		8. CONTRACT OR GRANT NUMBER(s) N00014-75-C-0678
9. PERFORMING ORGANIZATION NAME AND ADDRESS School of Operations Research and Industrial Engineering, Cornell University, Upson Hall Ithaca, New York 14853		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS Operations Research Program Office of Naval Research Arlington, VA 22217		12. REPORT DATE July 1977 13. NUMBER OF PAGES 131
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

cores	coalition formation	bargaining set
stable sets	abstract games	Caplow's theory of coalition formation
n-person games	dynamic solution	Gamson's theory of coalition formation
game theory	Shapley value	simple games

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The theory of n-person cooperative games presented by von Neumann and Morgenstern is a mathematical theory of coalition behaviour. A fundamental problem posed in game theory is to determine what outcomes are likely to occur if a game is played by "rational players". Given an n-person cooperative game and assuming rational behaviour, it is natural to inquire (1) which of the possible coalitions can be expected to form and (2) what will be the final payoffs to each of the players. However, most of the research in game theory has been concerned next page

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explicitly with predicting players' payoff and only implicitly (if at all) with predicting which coalitions shall form. In this investigation, the primary emphasis is on the first aspect of coalition behaviour, namely the formation of coalitions.

In one approach, the formation of coalitions is modelled as an abstract game. There are several solution concepts defined for abstract games such as the von Neumann-Morgenstern stable sets, the core due to Gillies and Shapley, and subsolutions due to Roth. A more descriptive solution concept reflecting the dynamic aspects of bargaining--called the dynamic solution--is proposed. The core and the dynamic solution are then used to analyze the abstract game formulation of the problem of coalition formation. The predictions of the abstract game models depend on the particular "payoff solution concept" used. I.e., the models assume that there is a rule governing the final payoff to each player as a function of the coalition structure that forms. The predictions of these models are then studied for the case of n-person cooperative games with side payments using various payoff solution concepts such as the individually rational payoffs, the core, the Shapley value and the bargaining set $M_1^{(i)}$.

In another approach, coalition formation is viewed as a bargaining process where the players are allowed to raise (coalitional) objections and (coalitional) counter objections in the same manner as in the Aumann-Maschler bargaining sets. While the Aumann-Maschler bargaining sets indicate distribution of joint payoffs given a fixed coalition structure, the restricted bargaining set proposed in this investigation indicates both formation of coalitions and distribution of payoffs as outcomes.

Coalition formation has been extensively studied by social scientists. Two classical theories of coalition formation--Caplow's theory of coalitions in the triad and Gamson's theory of coalition formation in weighted majority games without dictators or veto players--are mathematically analyzed and compared with the predictions of the abstract game models under the same assumptions.

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ITHACA, NEW YORK

14 TR-342

9 TECHNICAL REPORT NO. 342

11 July 1977   12 144 p.

6 ON GAME THEORY AND COALITION FORMATION

by

10 Prakash P. Shenoy

15 N00014-75-C-0678, NSF-MPS-75-02024

This research was supported in part by the Office of Naval Research under Contract N00014-75-C-0678, and the National Science Foundation under Grants MPS75-02024 and MCS77-03984.

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ON GAME THEORY AND COALITION FORMATION

Prakash Pundalik Shenoy, Ph.D.
Cornell University, 1977

The theory of n-person cooperative games presented by von Neumann and Morgenstern is a mathematical theory of coalition behaviour. A fundamental problem posed in game theory is to determine what outcomes are likely to occur if a game is played by "rational players". Given an n-person cooperative game and assuming rational behaviour, it is natural to inquire (1) which of the possible coalitions can be expected to form and (2) what will be the final payoffs to each of the players. However, most of the research in game theory has been concerned explicitly with predicting players' payoff and only implicitly (if at all) with predicting which coalitions shall form. In this investigation, the primary emphasis is on the first aspect of coalition behaviour, namely the formation of coalitions.

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ACKNOWLEDGMENTS

I would like to express my deepest gratitude to my thesis advisor, William F. Lucas. His guidance, constant encouragement and numerous suggestions provided an invaluable contribution to this work. In addition, he has been a friend, philosopher and guide in the role of the Chairman of my special committee during my entire graduate career at Cornell.

I am also grateful to Louis J. Billera for many helpful discussions and comments. Special thanks are due to Philip D. Straffin for pointing out several useful references in the social sciences and to Shigeo Muto for his comments on this work.

I give my thanks to Lionel I. Weiss and David Gries for serving on my special committee. Also, I would like to take this opportunity to express my deep appreciation to Professor M. N. Vartak, Head of the Department of Mathematics at Indian Institute of Technology, Bombay for having introduced me to the field of Operations Research. I acknowledge too the countless students and professors who have contributed to my education during the past two decades.

My graduate career would not have been possible without the generous financial support provided by teaching assistantships from the School of Operations Research and Industrial Engineering at Cornell, Fellowships from the Graduate School at Cornell and research assistantships on grants from the Office of Naval Research and the National Science Foundation.

I am fortunate to have had the fine skills of Kathy King, who did such an excellent job of typing the manuscript.

TABLE OF CONTENTS

LIST OF NOTATIONS

A(D): arc set of the digraph D

A-M: Aumann-Maschler

B_n: a pure bargaining game on n players

$\mathcal{B}$: a balanced set

C: the core of an abstract game

Co: the core of a game with side payments

$\mathrm{Conv}\{a_1,\ldots,a_p\}$: convex hull of the points in $\{a_1,\ldots,a_p\}$

c.s.: coalition structure

D: a digraph

D': converse digraph of digraph D

D*: condensation digraph of digraph D

dom: domination, a binary relation

Dom(x), $x \in X$: dominion of outcome x

$\mathrm{Dom}^k(x)$, $k > 1$: $\mathrm{Dom}(\mathrm{Dom}^{k-1}(x))$ where $\mathrm{Dom}^1(x) = \mathrm{Dom}(x)$

$\mathrm{Dom}^{-1}(x)$: inverse dominion of outcome x

t-dom: transitive domination, a binary relation

$\mathrm{dom}_R(S)$: domination via coalition R with respect to payoff solution concept S, a binary relation defined on the set $\Pi(S)$

dom(S): domination with respect to payoff solution concept S, a binary relation defined on the set $\Pi(S)$

w-dom(S): weak domination, a binary relation defined on the set $\Pi(S)$

s-dom(S): strong domination, a binary relation defined on the set $\Pi(S)$

E^k: Euclidean space of dimension k

2^{E^k}: set of all subsets of k dimensional Euclidean space E^k

G^N: set of all cooperative games with side payments on N, a Euclidean space of dimension $2^n - (n+1)$

g: $\min\limits_{R \in \mathcal{W}} \sum\limits_{i \in R} a_i$

I: individually rational payoffs, a payoff solution concept

iff: if and only if

$J_0(\mathcal{S})$: the core of the abstract game $(SC(\mathcal{S}), \mathrm{dom})$

$J_1(\mathcal{S})$: the dynamic solution of the abstract game $(SC(\mathcal{S}), \mathrm{dom})$

$K_0(\mathcal{S})$: the core of the abstract game $(\Pi(\mathcal{S}), \mathrm{dom}(\mathcal{S}))$

$K_1(\mathcal{S})$: the dynamic solution of the abstract game $(\Pi(\mathcal{S}), \mathrm{dom}(\mathcal{S}))$

M_n: a straight majority game on n players

$M_{n,k}$: a symmetric monotonic simple game on n players given by $R \in \mathcal{W} \iff |R| \geq k$

$M_1^{(i)}$: the Aumann-Maschler bargaining set, a payoff solution concept

$M_{1,\varepsilon}^{(i)}$: the ε-bargaining set

M_c: the coalitional bargaining set

M_r: the restricted bargaining set

$N = \{1,\ldots,n\}$: set of n players

2^N: set of all subsets (coalitions) of N

P: the dynamic solution of an abstract game

$\mathcal{P} = (P_1,\ldots,P_m)$: a partition of N into subsets $P_1,\ldots,P_m$

p.s.c.: payoff solution concept

p.c.: payoff configuration

q: quota of a weighted majority game

R: a subset (coalition) of N

$\mathcal{S}: \Pi \to 2^{E^n}$: a payoff solution concept

S: an elementary dynamic solution

$SC(\mathcal{S})$: set of all solution configuration with respect to p.s.c. $\mathcal{S}$

s.c.: solution configuration

s.t.: such that

s: $\min_{R \in \mathcal{W}} \sum_{i \in R} \phi_i(v)$

$T_1, \ldots, T_m$: strong components of digraph D

t: $\min_{R \in \mathcal{W}} |R|$

$v: 2^N \to E^1$: the characteristic function of a cooperative game with side payments

$v: 2^N \to E^n$: the characteristic function of a cooperative game without side payments

$\hat{v}$: the superadditive cover of the game v

$v|R$, $R \subset N$: the subgame on R

V(D): vertex set of the digraph D

V: a von Neumann-Morgenstern stable set of an abstract game

vN-M: von Neumann-Morgenstern

$w: \Pi \to E^1$: $w(P) = \sum_{P_j \in P} v(P_j)$ is the worth of partition P

$\mathcal{W}$: set of all winning coalitions

$\mathcal{W}^m$: set of all minimal winning coalitions

w.r.t.: with respect to

X: set of outcomes of an abstract game

$x(R)$, $x \in E^n$, $R \subset N$: denotes $\sum_{i \in R} x_i$

z: $\max_{P \in \Pi} w(P)$

$\alpha: N \to N$: a permutation function (one-one and onto)

$\beta(P)(i)$: number of players which player i controls in coalition structure P

$\emptyset$: the empty set

δ_R, $R \in \mathcal{B}$: weight of coalition R in a balanced collection

γ: Gamson power index, a payoff solution concept

κ: Caplow power index, a payoff solution concept

Π: set of all partitions of N

$\Pi(\mathcal{S})$: set of all viable partitions of N with respect to the payoff solution concept $\mathcal{S}$

Π_z: $\{\mathcal{P} \in \Pi: w(\mathcal{P}) = z\}$

Π_s: $\{\mathcal{P} \in \Pi$: $\mathcal{P}$ contains a coalition R such that $\sum_{i \in R} \phi_i(v) = s\}$

Π_g: $\{\mathcal{P} \in \Pi$: $\mathcal{P}$ contains a cheapest winning coalition, i.e., a coalition R such that $\sum_{i \in R} a_i = g\}$

Π_t: $\{\mathcal{P} \in \Pi$: $\mathcal{P}$ contains a winning coalition of size $t\}$

Φ: the Aumann-Dreze generalization of the Shapley value

Φ': the Gamson generalization of the Shapley value

$\phi(v) = (\phi_1(v),\ldots,\phi_n(v))$: the Shapley value of the game v corresponding to the grand coalition

$\sigma: \Pi \to E^n$: a payoff value concept, a payoff solution concept that, for every game, associates a unique vector with each partition of N

Γ: a game

Γ_z: a game (N,v_z) derived from the game (N,v) given by

$v_z(N) = z$, $v_z(R) = v(R)$ for all other $R \subset N$

$\leftarrow$: accessible, a binary relation

$\leftrightarrow$: communication, an equivalence relation

$\forall$: for each

$\exists$: there exists

$\Rightarrow$: implies

($\Rightarrow$): necessity

($\Leftarrow$): sufficiency

$\Leftrightarrow$: if and only if

$\square$: end of proof

ϵ: belongs to

$|R|$: cardinality of the set R

CHAPTER I

AN INTRODUCTION TO THEORIES OF COALITION FORMATION

1.1 A Statement of the Problem

The theory of n-person cooperative games presented by von Neumann and Morgenstern is a mathematical theory of coalition behaviour. A fundamental problem posed in game theory is to determine what outcomes are likely to occur if a game is played by "rational players". Given an n-person cooperative game in characteristic function form and assuming the players to be "rational", it is natural to inquire (1) which of the possible coalitions can be expected to form and (2) what will be the final disbursement of payoffs among the players. These two aspects of coalition behaviour are closely related. The final disbursement of payoffs among the players depend on the coalitions that finally form, and the coalitions that finally form depend on the available payoffs to each player in each of these coalitions. Since the publication in 1944 of the monumental work Theory of Games and Economic Behaviour [75] by von Neumann and Morgenstern, most of the research in n-person game theory has been concerned explicitly with predicting players' payoffs and only implicitly (if at all) with predicting which coalitions shall form. In this investigation, the primary emphasis is on the first aspect of coalition behaviour, namely the formation of coalitions. Several theories of coalition formation are proposed based on the theory of n-person games. As in most of n-person game theory, our models are (conditionally) normative and use only endogenous arguments, that is, only information contained in the characteristic function is used.

1.2 Game Theory and Coalition Behaviour

Games in characteristic function form were first considered explicitly and in detail in 1944 by von Neumann and Morgenstern (vN-M) [75]. Their theory of behaviour in cooperative situations is predicated on two assumptions. First, it is assumed that each coalition R of players can assure itself of a particular amount $v(R)$ of resource, independently of what the remaining players do. Second, it is assumed that any coalition may divide what it receives among its players in a completely arbitrary manner (in other words, there is no restriction on side payments between players).

vN-M also proposed the concept of "stable sets" as solutions to a game. The basic feature of this solution concept is the idea of dominance as a preference relation on the set of all "imputations". An imputation represents an allocation of the available resources among the n players acting as one coalition. However, on strictly mathematical grounds the theory of stable sets contains some unpleasant results. Shapley [90,91] and Lucas [55,57,60,63] exhibited a number of games with particularly pathological stable set solutions and Lucas [58,59] gave an example of a game with no stable sets settling a long standing conjecture on the question of existence of stable sets for all games. Besides these mathematical pathologies, Weber [112, pp. 4-6] indicates a number of philosophical objections to the concept of a stable set. In an approach to one of these objections, Vickrey [108] proposed the concept of "self-policing" sets of imputations, and investigated the existence of stable sets with the self-policing property. Another approach dates back to a suggestion of Nash [73] where he writes that:

> "A...type of application is to the study of cooperative games... . One proceeds by constructing a model of the pre-play negotiations so that the steps of the negotiation become moves in a larger non-cooperative game describing the total situation...thus the problem of analyzing a cooperative game becomes the problem of obtaining a suitable, and convincing,...model for the negotiation."

In recent years, the noncooperative approach to the cooperative games has been pursued by Selten [88], Harsanyi [39,41,44] and Weber [112]. In another approach, Roth [84] introduces some alternate solution concepts having some similarities to the vN-M stable sets. He is then quite successful in obtaining existence theorems for these concepts. (Cf. Weber [112, pp. 2-4] and Lucas [65].)

One of the simplest solution concepts is the core. The theory of the core is implicit in the theory of stable sets since the core is a subset of any stable set. The core was first studied explicitly however in the mid 1950's by Gillies [35] and Shapley. The core may sometimes be empty. Bondareva [19,20] and Shapley [93] give necessary and sufficient conditions for the nonemptiness of the core of an n-person game with side payments. Even when the core is nonempty, it may be "too small", as in "simple games with veto players" where it assigns all the payoff to the set of veto players even though they may not be "dictators", that is, they may need the help of some others to achieve these payoffs. (Cf. Lucas [62, p. 14.])

Several authors have proposed different value theories for cooperative games. A player's value is an a priori measure of his expected gain in a given game. Most value theories determine a unique imputation as their solution set. This outcome is justified by arguments based upon some concept of fairness as determined by certain axioms, upon some bargaining procedure

or arbitration scheme, or upon probabilistic considerations. The Shapley value [87] and the Banzhaf value [12] have been studied extensively and used in applications. These value concepts have been interpreted in several ways by Dubey [30], Davis [27], Myerson [70], Owen [76], Shapley and Shubik [94], Straffin [104], Roth [85], and others. Lucas [64] and Straffin [103] present a collection of various applications of these value concepts to real life situations.

All the solution concepts discussed so far describe the disbursement of payoffs among the players assuming that the grand coalition of all players forms. This idea is embodied in the definition of the imputation which has been involved in all the concepts so far discussed, namely the vN-M stable sets, the core and the value theories. However, Aumann and Dreze [8] present natural generalization of these solution concepts to a given partition of the set of all players.

Aumann and Maschler [9] introduced several somewhat similar solution concepts called bargaining sets. These sets describe what payoff vectors are "stable" once a given coalition structure (partition of the player set N into subsets) has formed. An individual outcome is stable in their sense if there is no "objection" to it or if each objection to it gives rise to a "counter objection". An individual outcome in a bargaining set can stand on its own, in contrast to an imputation in a vN-M stable set. In the vN-M theory it is the whole set which possesses a global stability or represents a standard of behaviour and not an individual imputation in this stable set. One of these bargaining sets, denoted by $M_1^{(i)}$, was shown by Peleg [79] to contain at least one payoff vector for each partitioning of the players into a coalition structure.

Davis and Maschler [28] introduced the kernel of a game which is always a nonempty subset of the bargaining set $M_1^{(i)}$.

Schmeidler [87] defined the nucleolus which turns out to be a unique outcome in the kernel and it is in the core if the latter is nonempty.

The bargaining sets, the kernel and the nucleolus describe outcomes associated with each coalition structure but they tell us nothing explicitly about which coalition structure(s) we could reasonably expect from rational players.

In addition to those mentioned above, several other solution concepts have been proposed. Some of these are reasonable outcomes due to Milnor [68], ψ-stability due to Luce [66, Ch. 10] and (k-r)-stability due to Shubik [99].

Milnor suggests three different systems of "reasonable" conditions, each of which isolates a subset of the set of all imputations. In doing so he has taken

> "...the point of view that it is better to have the set too large rather than too small. Thus it is not asserted that all these points within one of our sets are plausible as outcomes; but only that points outside these sets are implausible." [68]

ψ-stability is a property of pairs (x,P) where x is an imputation and P is a coalition structure given that P can only break up into certain other coalition structures. If the admissible coalition changes are specified by a rule $\psi(P)$, stating all the coalition structures which can form from any given P, then (x,P) is ψ-stable if none of the admissible coalitions can get more than its members get in (x,P).

A payoff vector is (k-r)-stable if, roughly speaking, no group of r players can do better using another strategy on the assumption that each of k players is committed to a "threat strategy" (i.e., will use a certain strategy whatever happens). (Cf. Taylor [105], pp. 363-364.)

In recent years, the dynamics of negotiation among the players have also been investigated. One approach to this problem concentrates on the use of discrete transfer schemes to study how players might arrive at a desirable outcome. A parallel approach employs systems of differential equations whose solutions represent a continuous transfer of payoff over time. The advantages of such an approach are multifold. Not only does it enable us to view game theory in terms of the actions of individuals or coalitions, but it also enables us to characterize solution concepts in terms of associated "behaviour". In 1968, Stearns [102] exhibited a sequence of discrete transfers of payoff among the players which converged to points in the kernel of Davis and Maschler [28]. In 1972, Billera [18] smoothed these transfer sequences to obtain a system of differential equations whose solutions represented a continuous transfer of payoff and which also converged to the kernel. In 1973, Kalai, Maschler and Owen [48] started a systematic investigation of asymptotically stable points in various bargaining sets. They also show that the nucleolus is a dynamically stable point for each system. Owen [78] proved that the conditions imposed by Kalai, Maschler and Owen [48] can be relaxed to a certain extent. In 1974, Wu [110] showed that a modification of the relaxation method of Agmon [1] could provide a discrete transfer sequence which converged to the core. Also, Wu and Billera [111] study a dynamic theory for the kernel given by Billera [18]. Grotte [36]

exhibits several systems of differential equations which represent possible behaviour patterns for the players. The solutions of these systems are shown to converge to a number of solution concepts, among them the core, the Shapley value, and in certain instances, the nucleolus. Maschler and Peleg [67] characterize all the dynamically stable points and the dynamically stable closed sets with respect to Stearns' system which belong to the appropriate bargaining sets. They are nucleoli of appropriate Liapunov functions. In particular, a new solution concept due to Gill Kalai, called the lexicographic kernel is shown to be a dynamically stable subset of the kernel.

In the vN-M characteristic function, a coalition is assigned a value which is assumed to be fixed irrespective of the organization of the remaining players into coalitions. In 1961, Thrall [106] formulated a theory of n-person cooperative games with side payments in terms of a partition function which is defined on the set of all partitions of the set of players (see Thrall and Lucas [107]). Thrall's formulation assigns a real numbered outcome to each coalition in each partition of the set of players. Subsequently Lucas [52,53,54,107] generalized and studied the concept of a vN-M stable set and some aspects of the core for a game in partition function form. Lucas [56] also exhibited a game in partition function form that has no stable sets.

In 1964, Eisenman [31,32] studied a natural generalization of games in partition function form which he calls "alliance games". Eisenman [32] also generalized the concept of Shapley value to alliance games.

Another generalization of the vN-M characteristic function model of a cooperative game deals with games in which side payments are either

forbidden altogether or are allowed but "utility is not transferable". We say utility is transferable if the increment to the payoff of a player caused by a transfer of money is proportional to the amount of money transferred (cf. Aumann [5]). Most solution concepts described so far for games with side payments have also been generalized to games without side payments. Aumann [10], Peleg [10,79] and Stearns [100,101] have studied the vN-M stable sets; Aumann [6], Billera [16,17] and Scarf [86] have analyzed the core; Asscher [3,4], Billera [15] and Peleg [80] have investigated the bargaining sets, and the value concept has been extended by Nash [72,74], Harsanyi [38,39,40], Isbell [45], Miyasawa [69], Shapley [92] and Owen [77]. Aumann [7] presents a survey of research on cooperative games without sidepayments up to 1967.

At this stage, it will be helpful to emphasize that n-person game theory as described so far has been concerned directly with the problem of disbursement of payoffs rather than the question of formation of coalitions. The one possible exception is implicit in the ψ-stability theory, in the sense that if some coalition structures cannot form stable pairs with any imputations, it could be surmised that such coalition structures will not be frequently observed (cf. Rapoport [82, p. 286]).

Recently Fink [33] has proposed a solution concept that

> "...yields assertions both on the coalition structures formed and on the distribution of the payoff among the players." [33]

Fink defines three dominance relations on the set of "individually rational payoff configurations" and studies the stable set solutions for these abstract games.

1.3 Theories of Coalition Formation in Sociology, Psychology and Political Science

In sociology, psychology and political science, there are a number of simple theories about the process of coalition formation. These theories consist essentially of an hypothesis concerning the player's goals or motives, a premise concerning their payoffs and an inference which singles out the coalitions most likely to form. We briefly describe some of these approaches in this section. However, we do not review a number of more informal and ad hoc theories of coalition formation nor the many experimental studies designed to test theories of coalition formation. Caplow [24] and Leiserson [51, Section 2.1] between them give a comprehensive coverage of these two topics.

Caplow's [22,23] theory of coalition formation is restricted to triads. A triad is a "three-person weighted majority game with a simple majority quota". Caplow's theory is based on the notion that

> "...the formation of given coalitions depends upon the initial distribution of power in the triad and, other things being equal, may be predicted to some extent when the initial distribution of power is known." [22]

He then describes six possible types of power distributions for groups of three players. On the basis of four assumptions which mainly postulate that a stronger member always controls a weaker one and that the goal is to dominate as many members as possible, he predicts, for each power distribution and as a function of the power rank order of the individual, the most probable coalitions to occur.

Following Caplow, Gamson [34] formulated a slightly more general theory of coalition formation in "proper" weighted majority games without

veto players or dictators. Gamson's main hypothesis is that

> "any participant will expect others to demand from a coalition a share of the payoff proportional to the amount of resources which they contribute to a coalition." [34]

Based on this assumption, he infers that a player will favor a "cheapest" winning coalition, i.e. a winning coalition whose total weight is a minimum among all winning coalitions.

Riker's [83] theory of coalition formation is applicable primarily to zero-sum games with side payments. Assuming rational behaviour and perfect information, he deduces that

> "...the equilibrium size of a winning coalition is always minimal." [83]

Leiserson [51] suggests several theories in which each player uses a "search strategy", looking for a coalition in a piecemeal, stepwise fashion which requires of him only 'local' rationality as opposed to 'global' rationality which requires the players to weigh their payoffs in every possible coalition. One of these search strategies attempts to minimize ideological diversity in the coalition; another takes into account differences on several issues and the possibilities of logrolling (cf. Taylor [105, p. 361]).

Axelrod's [11] theory of coalition formation is based on the notion that players tend to minimize conflict of interest. His main hypothesis is that

> "in a parliamentary democracy in which the parties can be placed in a one-dimensional ordinal policy space, minimal connected winning coalitions:

1) are likely to form more often than would be indicated by chance (even compared to just other winning coalitions), and
2) once formed are likely to be of longer duration than other coalitions." [11]

By connected coalitions, he means coalitions which consists of ideologically adjacent parties.

In recent years, Cross [26], Komorita and Chertkoff [49] and Komorita [50] have also proposed additional theories of coalition formation.

1.4 A Summary of Research

In this section we briefly summarize the research in this thesis.

In Chapter 2, a new solution concept, called the dynamic solution, based on the elementary theory of Markov chains, is defined for abstract games. The structure and properties of the solution are studied. The (payoff) dynamic solution of all 3-person games with side payments is determined. Finally, many games, pathological in their behaviour with respect to the classical von Neumann-Morgenstern theory of stable sets, are shown to be amenable to this approach.

In Chapter 3, we propose some theories of coalition formation based on the theory of n-person cooperative games. The predictions of these theories depend on the particular "payoff solution concept" used, i.e., the theories assume that there is a rule governing the distribution of payoffs to each player in each coalition structure. The various theories proposed are compared. Section 3.3 contains a representation of the problem and solutions by means of digraphs. In Sections 3.4-3.7, the predictions of the theories are characterized for the case of games with side payments using various payoff solution concepts such as the

individually rational payoffs, the core, the Shapley value and the bargaining set $M_1^{(i)}$. Finally in Section 3.8, some modifications of these theories are discussed.

In Chapter 4, Caplow's and Gamson's theories of coalition formation are mathematically interpreted, analyzed and compared with the models in Chapter 3.

In Chapter 5, some modifications of the Aumann-Maschler (A-M) bargaining set $M_1^{(i)}$ are discussed. The A-M bargaining set theory was developed to attack the following general question: If the players in a cooperative n-person game have decided upon a specific coalition structure, how then will they distribute among themselves the values of the various coalitions in such a way that some stability requirements will be satisfied (cf. Davis and Maschler [28, p. 39]). In our theory, we do *not* assume that players have any a priori preference for any particular coalition structure. Some examples, illustrating the basic differences between the A-M bargaining set $M_1^{(i)}$ and our restricted bargaining set M_r, are exhibited. A few general results are also presented.

Finally in Chapter 6, we present a brief summary of the research in this thesis and discuss its potential significance. Some questions left unanswered in this work are also raised.

CHAPTER II

A DYNAMIC SOLUTION CONCEPT FOR ABSTRACT GAMES

2.1 Introduction

Most solution concepts for n-person cooperative games are normative or prescriptive theories. A more descriptive theory reflecting the dynamic aspects of bargaining is proposed in this chapter. Section 2.2 contains some notation and definitions. We introduce two additional binary relations which depend on the binary relation domination in an abstract game. An interpretation of these relations is also presented. In Section 2.3, the concepts of an elementary dynamic solution and a dynamic solution are introduced and discussed. The properties of the dynamic solution are studied in Section 2.4. For an abstract game with a finite number of outcomes, the concept of dynamic solution coincides with a concept of an R-admissible set defined by Kalai, Pazner and Schmeidler in [46][†]. In Section 2.5 the dynamic solution for all 3-person games with side payments is determined. Finally in Section 2.6, many games, which have pathological behaviour in the classical von Neumann-Morgenstern theory of stable sets, are shown to be amenable to our approach.

2.2 Notations and Definitions

An abstract game is a pair (X,dom) where X is an arbitrary set whose members are called outcomes of the game, and dom is an arbitrary

[†] Kalai and Schmeidler [47] have also defined a solution concept similar to the dynamic solution. However, the research presented in this chapter was done independently of both these references.

binary relation defined on X and is called domination. An outcome $x \in X$ is said to be accessible from an outcome $y \in X$, denoted by $x \leftarrow y$ (or $y \rightarrow x$), if there exists outcomes $z_0 = x, z_1, z_2, \ldots, z_{m-1}, z_m = y$, where m is a positive integer, such that

$$(2.1) \qquad x = z_0 \text{ dom } z_1 \text{ dom } z_2 \text{ dom } \ldots \text{ dom } z_{m-1} \text{ dom } z_m = y.$$

Also assume $x \leftarrow x$, i.e. an outcome is accessible from itself. Clearly, the binary relation accessible is transitive and reflexive.

An interpretation of the relation accessible is as follows: If the players are considering an outcome y at some stage, then an outcome they will consider next will be a $z \in X$ such that z dom y. If $x \leftarrow y$ and if the players are considering outcome y at some time, then it is possible that they will consider outcome x at some future time. I.e. one may interpret the relation as a possible succession of transitions from one outcome to another.

Two outcomes x and y which are accessible to each other are said to communicate and we write this as $x \leftrightarrow y$. Since the relation accessible is transitive and reflexive it follows that

Proposition 2.1. Communication is an equivalence relation.

We can now partition the set X into equivalence classes. Two outcomes are in the same equivalence class if they communicate with each other. We say that the abstract game is irreducible if this equivalence relation induces only one class. The set

$$\text{(2.2)} \qquad \mathrm{Dom}(x) = \{y \in X\colon x \text{ dom } y\}$$

is called the <u>dominion</u> of x. Similarly we define the dominion of any subset $A \subset X$ by

$$\text{(2.3)} \qquad \mathrm{Dom}(A) = \bigcup_{x \in A} \mathrm{Dom}(x)$$

Also define the <u>inverse dominion</u> of x by

$$\text{(2.4)} \qquad \mathrm{Dom}^{-1}(x) = \{y \in X\colon y \text{ dom } x\}.$$

The <u>core</u> C (due to Gillies [35]) of an abstract game is defined to be the set of undominated outcomes. I.e.

$$\text{(2.5)} \qquad C = X - \mathrm{Dom}(X).$$

We can rewrite the definition of the core in terms of the relation accessible as follows:

$$\text{(2.6)} \qquad C = \{x \in X\colon \text{ For all } y \in X,\ y \neq x, \text{ we have } y \not\to x\},$$

i.e., in the terminology of Markov chains, the core is the set of all <u>absorbing</u> outcomes. Note that each outcome in the core (if nonempty) is an equivalence class by itself.

A <u>vN-M stable set</u> V (due to von Neumann and Morgenstern [75]) of an abstract game is any $V \subset X$ such that

$$\text{(2.7)} \qquad V = X - \mathrm{Dom}(V).$$

Any vN-M stable set V satisfies internal stability and external stability, i.e.,

(2.8) $$V \cap \mathrm{Dom}(V) = \emptyset \quad \text{and} \quad V \cup \mathrm{Dom}(V) = X.$$

In recent years, Behzad and Harary [13,14] and Shmadich [98] have characterized finite abstract games for which vN-M stable sets exist.

2.3 The Dynamic Solution

We define an elementary dynamic solution (elem. d-solution) of the abstract game (X,dom) as a set $S \subset X$ such that

(2.9) $$\text{if } x \in S,\ y \in X\text{-}S, \text{ then } y \not\leftarrow x \text{ and}$$

(2.10) $$\text{if } x,y \in S, \text{ then } x \leftarrow y \text{ and } y \leftarrow x.$$

Condition (2.9) requires S to be 'externally stable' in a dynamic sense, i.e. if the players are considering $x \in S$ at some time, then they will never consider any outcome that is not in S in the future. We can think of Condition (2.10) as 'internal stability' in a dynamic sense. I.e., if the players make a transition (in the consideration of outcomes) from x to y then it is possible that the players will again consider the outcome x in the future.

Proposition 2.2. An elem. d-solution S is an equivalence class.

Proof: By Condition (2.10), S is contained in an equivalence class H, i.e. $S \subset H$. Suppose $S \neq H$. Let $x \in H\text{-}S$ and $y \in S$. Then $x \leftarrow y$ since H is an equivalence class, which contradicts (2.9).

The converse, however, is not always true, i.e., an equivalence class need not be an elem. d-solution. Condition (2.9) requires S to be (in the terminology of Markov chains) a non-transient (recurrent, persistent) equivalence class.

Proposition 2.3. Each outcome in the core C of the game is an elem. d-solution.

The proof follows from the definition of the core in (2.6).

The dynamic solution (d-solution) P of the game is the union of all distinct elementary dynamic solutions. I.e.

(2.11) $$P = \cup\{S \subset X: S \text{ is an elem. d-solution}\}.$$

We can interpret P as the set of all likely outcomes of the game.

Proposition 2.4. For any abstract game, the dynamic solution always exists and is unique.

Proof: Existence follows from the fact that the empty set $\emptyset$ is always an elem. d-solution. Uniqueness is clear from Proposition (2.2) and the definition of the d-solution.

Proposition 2.5. $C \subset P$

The proof follows from Proposition 2.3 and the definition of P.

2.4 Properties of the Dynamic Solution

If X is a finite set, then our definition of the d-solution coincides with the definition of the R-admissible set due to Kalai, Pazner and

Schmeidler [46]. In this section we demonstrate the equivalence of the two definitions. This will also illustrate some of the properties of the d-solution.

Lemma 2.6. If X is a finite set, then P is the d-solution if and only if P satisfies:

(2.12) For all $x,y \in P$, $y \leftarrow x \Longleftrightarrow x \leftarrow y$.

(2.13) If $x \in P$, $y \in X\text{-}P$, then $y \not\leftarrow x$. And

(2.14) if $y \in X\text{-}P$, then $\exists\ x \in P$ such that $x \leftarrow y$.

Proof: ($\Longrightarrow$) It is clear from the definition of P that it satisfies Conditions (2.12) and (2.13). Suppose Condition (2.14) does not hold. Then for some $y_1 \in X\text{-}P$, $x \not\leftarrow y_1$ for all $x \in P$. Let $A_1(y_1) \subset X\text{-}P$ be the equivalence class containing y_1. If $A_1(y_1)$ satisfies Condition (2.9), then $A_1(y_1)$ is an elem. d-solution which is a contradiction. If not, then $\exists\ y_2 \in X\text{-}P\text{-}A_1(y_1)$ such that $y_2 \leftarrow x$ for some $x \in A_1(y_1)$. Let $A_2(y_2) \subset X\text{-}(P \cup A_1(y_1))$ be the equivalence class containing y_2. Repeating this argument, since X is finite, we get an equivalence class $A_k(y_k) \subset X - P - \bigcup_{i=1}^{k-1} A_i(y_i)$ satisfying Condition (2.9). Hence $A_k(y_k)$ is an elem. d-solution, which is a contradiction!

($\Longleftarrow$) Statements (2.12) and (2.13) imply that P is a union of elem. d-solutions. Suppose some elem. d-solution S is not included in P, and let $y \in S \subset X\text{-}P$. Then from Condition (2.14) $\exists\ x \in P$ such that $x \leftarrow y$. But $x \notin S$ contradicts the fact that S is an elem. d-solution! Hence P is the union of all elem. d-solutions. $\square$

<u>Theorem 2.7</u>. If X is a finite set, then the d-solution is nonempty and unique.

<u>Proof</u>: Nonemptiness follows from Condition (2.14) of Lemma 2.6. Uniqueness follows from Proposition 2.4.

<u>Remark</u>: If R is an arbitrary binary relation, Kalai, Pazner and Schmeidler define an R-admissible set as a subset of X satisfying Conditions (2.12), (2.13) and (2.14) with the binary relation R substituted in place of $\leftarrow$.

Define a binary relation <u>transitive-domination</u> denoted by t-dom as follows:

(2.15) For all $x,y \in X$, x t-dom $y \iff x \leftarrow y$ and $y \not\leftarrow x$.

Transitive domination is irreflexive and transitive. The following lemma is proved in Kalai, Pazner and Schmeidler [46].

<u>Lemma 2.8</u>. If X is a finite set, the d-solution P satisfies:

(2.16) For all $x,y \in P$, x t$\not$dom y and y t$\not$dom x (internal stability).

(2.17) For all $y \in X\text{-}P$, $\exists\, x \in P$ such that x t-dom y (external stability).

I.e. P is the unique vN-M stable set and the core of the abstract game (X,t-dom).

The following results are easy consequences of the definition of the d-solution. Nevertheless, they are useful in computing the d-solution.

Proposition 2.9. If $x,y \in X$ such that $x \leftarrow y$ and $y \not\leftarrow x$, then $y \notin P$.

Proof: If $x \in P$, then $y \in P$ contradicts Condition (2.12). If $x \notin P$, then $y \in P$ contradicts Condition (2.13).

Corollary 2.10. Let y be an outcome that is not in the core. Then $\mathrm{Dom}(y) = \emptyset \Rightarrow y \notin P$.

Proposition 2.11. $x \notin P \Rightarrow (\bigcup_{k=1}^{m} \mathrm{Dom}^k(x)) \cap P = \emptyset$ for all integers $m \geq 1$.

Proof: $y \in \bigcup_{k=1}^{m} \mathrm{Dom}^k(x) \Rightarrow y \rightarrow x$ for $x \notin P \Rightarrow y \notin P$.

Proposition 2.12. If the core C is the unique vN-M solution, then $P = C$.

Proof: From Proposition 2.5, $C \subset P$. Since C is the unique vN-M stable set, $y \in X\text{-}C \Rightarrow \exists\, x \in C$ such that $x \leftarrow y$. But $y \not\leftarrow x$ (since $x \in C$). Hence $y \notin P$ (by Proposition 2.9).

Corollary 2.13. Let C be a nonempty core. If $y \in \mathrm{Dom}^k(C)$ for some integer $k \geq 1$ then $y \notin P$. I.e. $P \subset X - \bigcup_{j=1}^{m} \mathrm{Dom}^j(C)$ for every integer $m \geq 1$.

2.5 Dynamic Solutions of 3-Person Games

A cooperative n-person game in characteristic function form is a pair (N,v) where $N = \{1,2,\ldots,n\}$ denotes the set of players and v is a non-negative real valued function defined on the subsets of N which satisfies $v(\emptyset) = 0$ and $v(\{i\}) = 0$ for all $i \in N$. The subsets of N are called coalitions. A coalition structure (c.s.) $\mathcal{P} = \{P_1,\ldots,P_m\}$ is a partition of N into disjoint (nonempty) coalitions. The set of (payoff) outcomes corresponding to coalition structure $\mathcal{P}$ is denoted

by $X(P)$, where

(2.18) $$X(P) = \{x \in E^n : x_i \geq 0 \text{ for all } i \in N \text{ and } \sum_{i \in P_j} x_i = v(P_j) \text{ for each } P_j \in P\}$$

The elements of the set $X(\{N\})$ are referred to as imputations. Domination is defined as follows:

> $x \in X(P)$ is said to dominate $y \in X(P)$ via coalition R, denoted by $x \text{ dom}_R\, y$ if $x_i > y_i$ for all $i \in R$ and $\sum_{i \in R} x_i \leq v(R)$.
> x dominates y, denoted by x dom y if $\exists$ a nonempty $R \subset N$ such that $x \text{ dom}_R\, y$.

In the abstract game $(X(P),\text{dom})$ as defined above, we cannot have domination via N and via one player coalitions. Also, if $x_i = 0$, then x does not dominate any other outcome via coalitions that contain player i.

Lemma 2.14. Let Γ be a 3-person game and P be a c.s. that contains only one-player or two-player coalitions. Then the dynamic solution of the game $(X(P),\text{dom})$ is the entire set of outcomes, i.e. $P(P) = X(P)$.

So we need concern ourselves with only the c.s. $P = \{N\}$. Let $C(\{N\})$ denote the core of the abstract game $(X(\{N\}),\text{dom})$. To condense notation we will denote $P(\{N\})$ and $C(\{N\})$ by $P(N)$ and $C(N)$ respectively. Assume without loss of generality that the characteristic function satisfies

(2.19) $$v(\{1,2\}) \le v(\{1,3\}) \le v(\{2,3\}).$$

Let $v(\{1,2\}) = a$, $v(\{1,3\}) = b$, $v(\{2,3\}) = c$ and $v(\{1,2,3\}) = d$.

The following inclusive cases should be distinguished:

Case 1) $d \ge (a+b+c)/2$, $d \ge c$.

In this case the core $C(N) \ne \emptyset$ and is given by

(2.20) $$C(N) = \{x \in E^3\colon x_i \ge 0 \text{ for all } i \in N,\ x_1 + x_2 \ge a,\ x_1 + x_3 \ge b,\ x_2 + x_3 \ge c \text{ and } x_1 + x_2 + x_3 = d\}.$$

The d-solution is given by $P(N) = C(N)$. (See Figure 2.1.)

Case 2) $d < (a+b+c)/2$, $d \ge c$.

In this case $C(N) = \emptyset$. The d-solution is given by

$$P(N) = \mathrm{Conv}\{w_1,w_2,w_3\} - \{w_1,w_2,w_3\} \text{ where}$$

$$w_1 = (a+b-d,\ d-b,\ d-a),$$

$$w_2 = (d-c,\ a+c-d,\ d-a),$$

$$w_3 = (d-c,\ d-b,\ b+c-d)$$

and $\mathrm{Conv}\{a_1,\dots,a_p\}$ denotes the convex hull of the points in $\{a_1,\dots,a_p\}$. (See Figure 2.2.)

Case 3) $a \le b \le d < c$, $d \ge a+b$.

In this case the core $C(N) \ne \emptyset$ and is given by

$$C(N) = \mathrm{Conv}\{(0,\ a,\ d\text{-}a),\ (0,\ d\text{-}b,\ b)\}$$

and the d-solution is given by $P(N) = C(N)$. (See Figure 2.3.)

Case 4) $a \leq b \leq d < c$, $d < a+b$.

In this case, $C(N) = \emptyset$. The d-solution is given by

$$P(N) = \mathrm{Conv}\{(a+b-d,\ d-b,\ d-a),\ (0,\ a,\ d-a),\ (0,\ d-b,\ b)\} - \{(a+b-d,\ d-b,\ d-a),\ (0,\ a,\ d-a),\ (0,\ d-b,\ b)\}$$

(See Figure 2.4.)

Case 5) $a \leq d < b \leq c$.

In this case $C(N) = \emptyset$. The d-solution is given by

$$P(N) = \mathrm{Conv}\{(a,\ 0,\ d-a),\ (0,\ a,\ d-a),\ (0,\ 0,\ d)\} - \{(a,\ 0,\ d-a),\ (0,\ a,\ d-a),\ (0,\ 0,\ d)\}$$

(See Figure 2.5.)

Case 6) $d < a \leq b \leq c$.

In this case $C(N) = \emptyset$. The d-solution is given by

$$P(N) = \mathrm{Conv}\{(d,\ 0,\ 0),\ (0,\ d,\ 0),\ (0,\ 0,\ d)\} - \{(d,\ 0,\ 0),\ (0,\ d,\ 0),\ (0,\ 0,\ d)\}$$

(See Figure 2.6.)

Thus all cases have been considered.

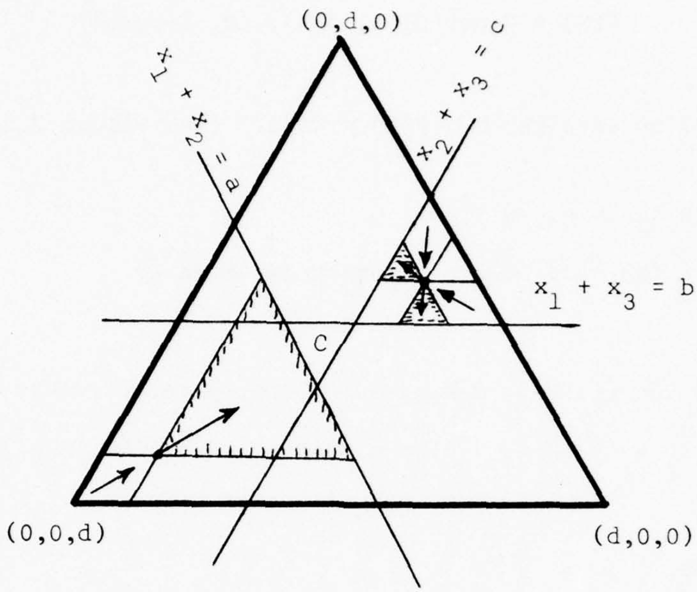

Figure 2.1. The dynamic solution P(N) of a 3-person game, Case 1). The arrows in the figure indicate transitions.

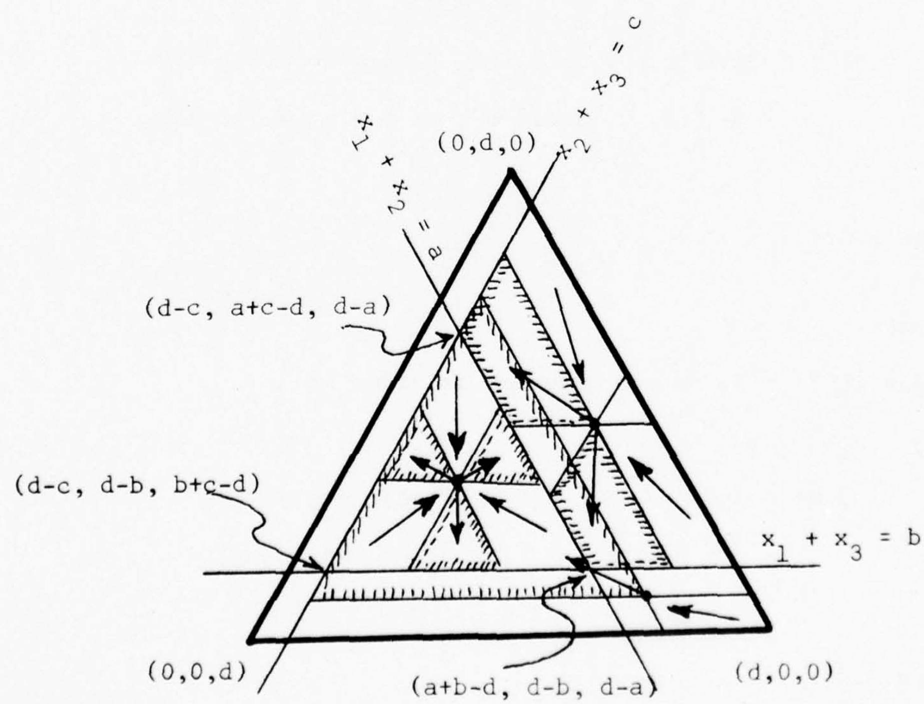

Figure 2.2. The dynamic solution P(N) of a 3-person game, Case 2).

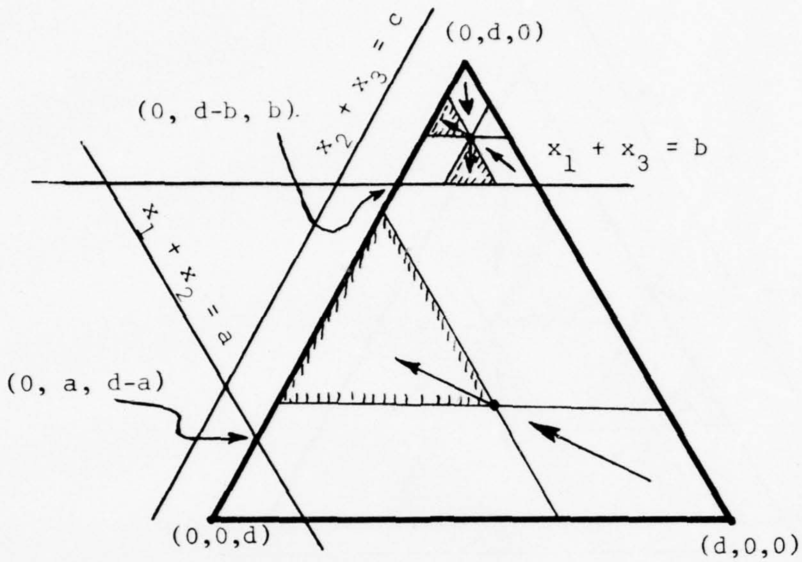

Figure 2.3. The dynamic solution P(N) of a 3-person game, Case 3).

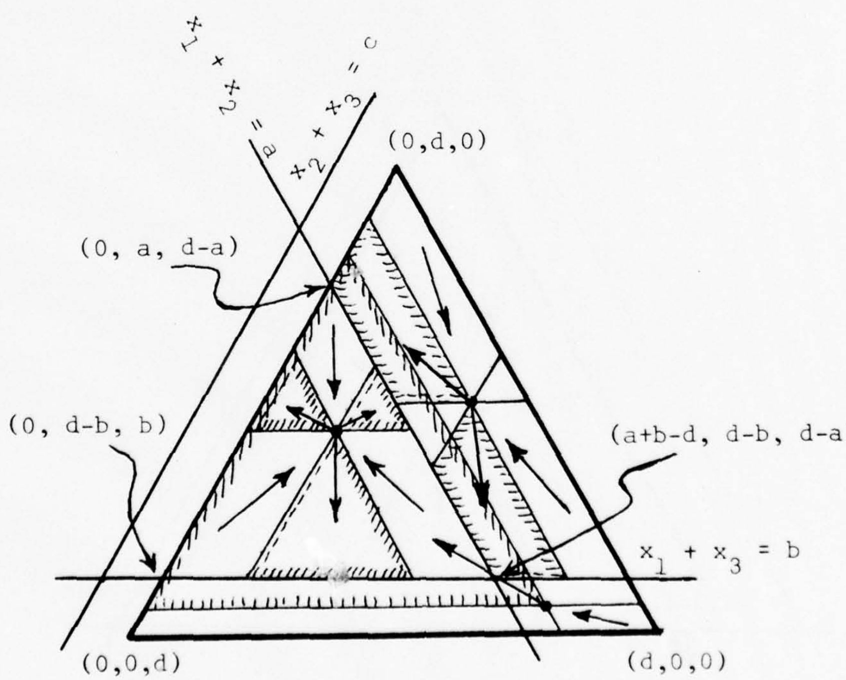

Figure 2.4. The dynamic solution P(N) of a 3-person game, Case 4).

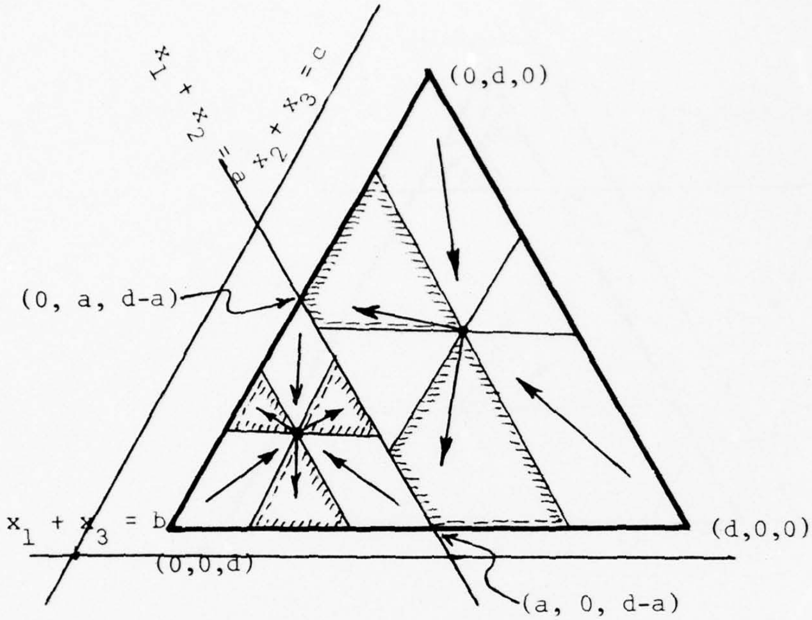

Figure 2.5. The dynamic solution P(N) of a 3-person game, Case 5).

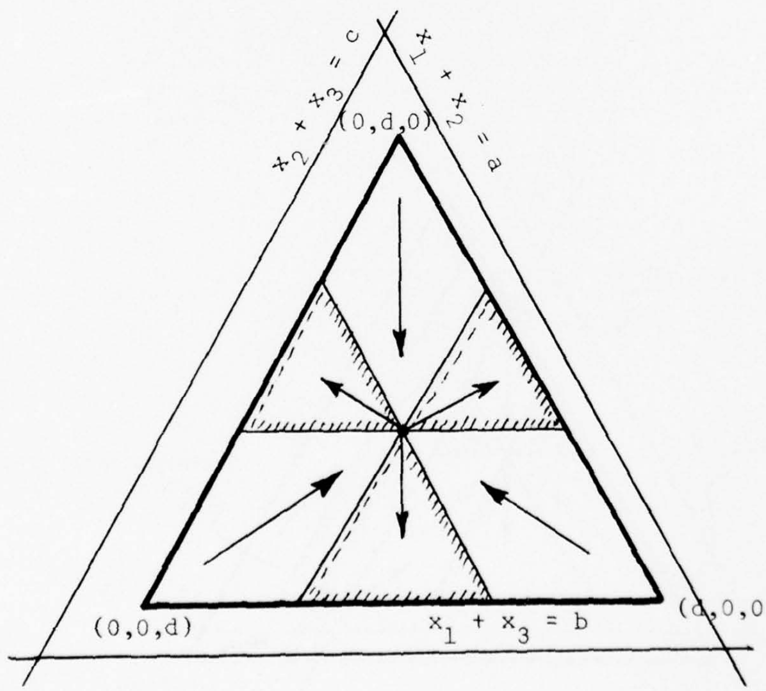

Figure 2.6. The dynamic solution P(N) of a 3-person game, Case 6).

2.6 Some Examples

In this section we study several examples which are pathological in their behaviour with respect to the classical vN-M theory of stable sets.

Example 2.1. (A 5-person game with a unique vN-M stable set strictly larger than the core. See Lucas [55].)
Consider the 5-person game given by

$$v(12345)^{\dagger} = 2, \quad v(12) = v(34) = v(135) = v(245) = 1,$$
$$v(R) = 0 \quad \text{otherwise.}$$

The core $C = \mathrm{Conv}\{(1,0,0,1,0), (0,1,1,0,0)\}$ and the unique vN-M solution is given by

$$V = \mathrm{Conv}\{(1,0,0,1,0), (1,0,1,0,0), (0,1,1,0,0), (0,1,0,1,0)\}.$$

The d-solution coincides with the core. This is seen as follows. We have $\mathrm{Dom}(C) = X - V$ and $V - C \subset \mathrm{Dom}(X-V)$. Hence $C = X - \bigcup_{j=1}^{2} \mathrm{Dom}^j(C)$. By Corollary 2.10, it follows that $P = C$.

Example 2.2. (A 5-person game with a unique stable set which is non-convex. See Lucas [57].)
Consider the 5-person game given by

†To condense notation we shorten expressions like $v(\{1,2,3,4,5\})$ to $v(12345)$.

$v(12345) = 3, \quad v(234) = v(345) = 2,$

$v(12) = v(45) = v(35) = v(34) = 1, \quad v(R) = 0$ otherwise.

For this game

$$X = \{x \in E^5 : \sum_{i \in N} x_i = 3, \quad x_i \geq 0 \text{ for all } i \in N\}.$$

Let

$$B = \{x \in X: \sum_{i \in R} x_i \geq v(R) \text{ for all } R \subset N \text{ except } \{2,3,4\}\}.$$

Then the core C of the game is given by

$$C = \{x \in B: x_2 + x_3 + x_4 \geq 2\}.$$

It can be easily shown that $\text{Dom}(C) \supset X - B$ and $B - C \subset \text{Dom}(X-B)$. Hence by Corollary 2.10, we have $P = C$.

Example 2.3. (A game with no symmetric stable set. See Lucas [57].) Let $N = \{1,\ldots,8\}$, $v(N) = 4$, $v(1357) = 3$, $v(257) = v(457) = 1$, $v(12) = v(34) = v(56) = v(78) = 1$, $v(R) = 0$ for all other $R \subset N$. For this game,

$$X = \{x \in E^8 : \sum_{i \in N} x_i = 4, \text{ and } x_i \geq 0 \text{ for all } i \in N\}.$$

Let

$$H = \{x \in X: x_1 + x_2 = x_3 + x_4 = x_5 + x_6 = x_7 + x_8 = 1\}.$$

Then the core C of the game is given by

$$C = \{x \in H: x_1 + x_3 + x_5 + x_7 \geq 3\}.$$

Define $F_i = \{x \in H: x_i = 1\}$ for $i = 1,3,5,7,$ and

$$F = F_1 \cup F_3 \cup F_5 \cup F_7 - C.$$

It is shown in Lucas [57] that $\mathrm{Dom}(C) = (X-H) \cup (H - (C \cup F))$. It is also clear that $(H-C) \subset \mathrm{Dom}(X-H)$. Hence $C = X - \mathrm{Dom}^2(C)$. By Corollary 2.10 it follows that $P = C$.

Example 2.4. (A game with no vN-M stable set. See Lucas [58,59].) Lucas [58] constructs a ten-person game in which the set of imputations can be partitioned into regions as follows:

$$X = (X-B) \cup (B - (C \cup E \cup F)) \cup (C \cup E \cup F)$$

where C is the nonempty core. The domination relations is such that

(1) $\mathrm{Dom}(C) \supset (X-B) \cup (B - (C \cup E \cup F))$,

(2) $F \cap \mathrm{Dom}(C \cup E \cup F) = \emptyset$,

(3) $E \subset \mathrm{Dom}(X-B)$.

By corollary 2.10 and Relation (1), $P \subset (C \cup E \cup F)$, Relation (2) $\Rightarrow F \subset \mathrm{Dom}(\{X-B\} \cup \{B - (C \cup E \cup F)\}) \Rightarrow F \cap P = \emptyset$ using Proposition 2.11 and Relation (3) $\Rightarrow E \cap P = \emptyset$ by Proposition 2.11. Hence $P = C$.

Example 2.5. (An 8-person game with a unique stable set that is non convex. See Lucas [60].)

Let $N = \{1,\ldots,8\}$, $v(N) = 4$, $v(1467) = 2$, $v(12) = v(34) = v(56) = v(78) = 1$, $v(R) = 0$ for all other $R \subset N$. For this game it can be shown as in Example 2.3 that $P = C$.

A game without side payments is a triple (N,v,X) where $N = \{1,\ldots,n\}$ is a set of n players, v is a "generalized characteristic function" and X is the set of imputations. A generalized characteristic function v maps nonempty subsets of N into subsets of n-dimensional space E^n, where the subset $v(R)$ assigned to coalition R consists of all vectors x such that R can guarantee all of its members at least their share in x. We assume that v satisfies the following axioms for any nonempty $R \subset N$.

(1) $v(R)$ is closed, nonempty and convex.

(2) If $x \in v(R)$ and $y_i \leq x_i$ for all $i \in R$ then $y \in v(R)$.

(3) $v(R_1) \cap v(R_2) \subset v(R_1 \cup R_2)$ whenever $R_1 \cap R_2 = \emptyset$.

(4) $x \in v(N) \Longleftrightarrow x_i \leq y_i$ for some $y \in X$ and for all $i \in N$.

Example 2.6. (A 7-person non side payment game with no vN-M stable sets. See Stearns [101].)

Let $N = \{1,\ldots,7\}$ and X be the convex hull of the five imputations

$$p^1 = (1,1,2,0,0,0,0) \qquad c = (2,0,2,0,2,0,1)$$

$$p^2 = (0,0,1,1,2,0,0) \qquad o = (0,0,0,0,0,0,0)$$

$$p^3 = (2,0,0,0,1,1,0)$$

Let the "minimal winning" coalitions be

$$(135),\ (127),\ (347),\ (567).$$

Note that a coalition is winning if it *contains* a minimal winning coalition as a subset. Define $v\colon 2^N - \emptyset \to E^7$ by

$$v(R) = \begin{cases} \{x \in E^7 : x_i \leq y_i \text{ for all } i \in R \text{ and for } \underline{\text{some}}\ y \in X\} & \text{when } R \text{ is winning} \\ \{x \in E^7 : x_i \leq y_i \text{ for all } i \in R \text{ and for } \underline{\text{all}}\ y \in X\} & \text{when } R \text{ is non-winning.} \end{cases}$$

The core of this game is the single imputation c. The d-solution is $P = C$. This is seen as follows.

$$\mathrm{Dom}(c) = X - (L_1 \cup L_2 \cup L_3) \quad \text{where} \quad L_i = [c, p^i]$$

the closed line segment joining c and p^i for $i = 1,2,3$. Let $x \in L_i - c$, i.e., $x = (\lambda+1,\ 1-\lambda,\ 2,\ 0,\ 2\lambda,\ 0,\ \lambda)$ for some $0 \leq \lambda < 1$. Let

$$y^1 = (2,\ 0,\ 2\lambda^1,\ 0,\ \lambda^1+1,\ 1-\lambda^1,\ \lambda^1) \quad \text{where} \quad \lambda < \lambda^1 < 1,$$

$$y^2 = (2\lambda^2,\ 0,\ \lambda^2+1,\ \lambda^2-1,\ 2,\ 0,\ \lambda^2) \quad \text{where} \quad \lambda^1 < \lambda^2 < 1 \quad \text{and}$$

$$y^3 = (\lambda^3+1,\ 1-\lambda^3,\ 2,\ 0,\ 2\lambda^3,\ 0,\ \lambda^3) \quad \text{where} \quad \lambda^2 < \lambda^3 < 1.$$

Then $y^3 \ \mathrm{dom}_{(127)}\ y^2 \ \mathrm{dom}_{(347)}\ y^1 \ \mathrm{dom}_{(567)}\ x$. Therefore $y^3 \leftarrow x$. But

$x \not\prec y^3$. Hence by Proposition 2.9, $x \notin P$. Hence $P = C = \{c\}$. (See Figure 2.7.)

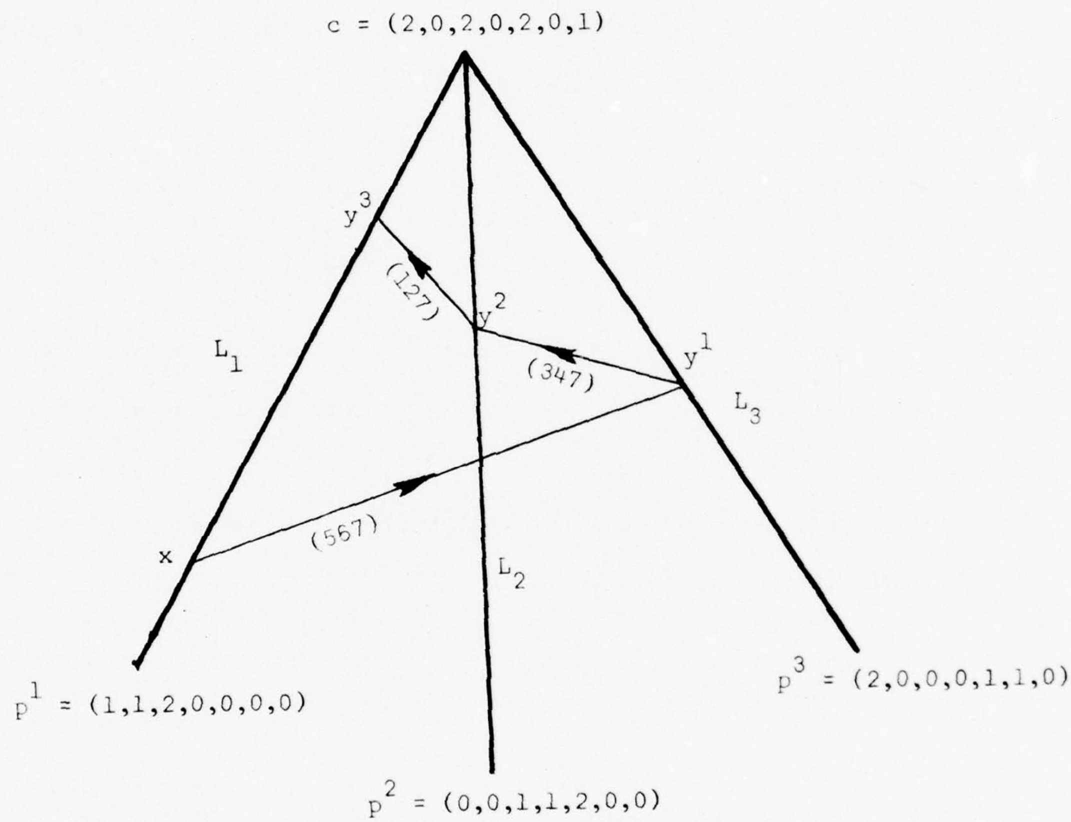

Figure 2.7. A 7-person non side payment game with no stable sets.

CHAPTER III

SOME THEORIES OF COALITION FORMATION

3.1 Introduction

In this chapter, we propose several theories of coalition formation. In one approach, coalition structures are modelled as the outcomes of an abstract game on which an appropriate domination relation is defined. In another approach, payoff disbursements and coalition structures are modelled as outcomes. In both cases, we study the core and the dynamic solution of the abstract game. The two models are then compared. Section 3.3 contains a representation of the models by means of digraphs. The core and the dynamic solution of the abstract game are then described in graph-theoretic terminology. In Sections 3.4-3.7, the solutions of the abstract games are characterized for the case of games with side payments using various payoff solution concepts such as the individually rational payoffs, the core, the Shapley value and the bargaining set $M_1^{(i)}$. Finally in Section 3.8, we discuss possible modifications in the definition of the domination relation in the case where coalition structures alone are modelled as outcomes.

3.2 The Models

We shall first introduce some notations and definitions. Let $N = \{1,\ldots,n\}$ denote the set of players. Let Γ be an n-person cooperative game (with side payments, without side payments or a game

in partition function form[†]). Let 2^N denote the set of all subsets (coalitions) of N and Π denote the set of all partitions (coalition structures) of N. Let $S: \Pi \to 2^{E^n}$ be a payoff solution concept, where 2^{E^n} denotes the set of all subsets of the n-dimensional Euclidean space E^n. Intuitively, given that the players in N align themselves into coalitions in the c.s. $P \in \Pi$, we interpret $S(P)$ as the set of all likely disbursements of payoffs to players in N. E.g. S may denote the individually rational payoffs, the core, a vN-M stable set, the Shapley value, the bargaining set $M_1^{(i)}$, the kernel, the nucleolus or any other payoff solution concept that indicates disbursement of payoffs as solutions of an n-person cooperative game. For $P \in \Pi$, $S(P)$ may be the empty set $\emptyset$ (e.g. the core), or a single point in E^n (e.g. the Shapley value or the nucleolus) or a nonempty subset of E^n (e.g. the bargaining set $M_1^{(i)}$ or the kernel). If $S(P) = \emptyset$ (interpreting this fact as players unable to reach an agreement on the disbursement of payoffs when they are aligned into coalitions in P), then we will assume for simplicity of exposition that P is not viable. Let $\Pi(S)$ denote the set of all viable coalition structures with respect to the payoff solution concept (p.s.c.) S, i.e.,

$$(3.1) \qquad \Pi(S) = \{P \in \Pi : S(P) \neq \emptyset\}.$$

Definition 3.1. A solution configuration with respect to p.s.c. S is *a pair (x,P) such that $x \in S(P)$ and $P \in \Pi(S)$.*

[†]All terms not defined in this text appear in the appendix.

A solution configuration w.r.t. p.s.c. S represents a possible outcome of the n-person cooperative game where S represents any appropriate payoff solution concept. Let $SC(S)$ denote the set of all solution configurations w.r.t. p.s.c. S, i.e.

$$(3.2) \qquad SC(S) = \bigcup_{P \in \Pi(S)} [S(P) \times \{P\}]$$

We now define a binary relation, domination, on the set $SC(S)$ as follows:

Definition 3.2. Let (x, P_1) and (y, P_2) belong to $SC(S)$. Then (x, P_1) dominates (y, P_2), denoted by (x, P_1) dom (y, P_2) iff

$$(3.3) \quad \exists \text{ a nonempty } R \in P_1 \text{ such that } x_i > y_i \text{ for all } i \in R.$$

Intuitively, if (x, P_1) dom (y, P_2), then the players in some coalition R in c.s. P_1 prefer P_1 to P_2. We require the players in subset R to be together in a coalition in c.s. P_1 so that there is no conflict of interest between these player's preference for P_1 and their allegiance to the other players in their coalition.

The dominance relation as defined above may be neither irreflexive nor transitive. We now have an abstract game $(SC(S), \text{dom})$ where $SC(S)$ is the set of outcomes and dom is a binary relation on $SC(S)$. For this abstract game, we look at the core and the dynamic solution as defined in Chapter 2.

Definition 3.3. Let Γ be an n-person cooperative game[†] and S be a p.s.c. The core of solution configurations w.r.t. p.s.c. S, denoted by $J_0(S)$, is the core of the abstract game $(SC(S), \text{dom})$.

Definition 3.4. Let Γ be an n-person cooperative game and S be a p.s.c. The dynamic solution of solution configurations w.r.t. p.s.c. S, denoted by $J_1(S)$, is the dynamic solution of the abstract game $(SC(S), \text{dom})$.

From Proposition 2.5, we obtain the following result.

Proposition 3.1. $J_0(S) \subset J_1(S)$.

The core of an abstract game is a very intuitive and plausible solution concept. However, for some games and for certain p.s.c., $J_0(S)$ may be an empty set. In such cases, we can proceed to look at $J_1(S)$ as a solution concept. If the p.s.c. S is such that $S(P)$ is a unique point in E^n for each $P \in \Pi(S)$ with $\Pi(S) \neq \emptyset$, then the set $SC(S)$ is finite and nonempty. By appealing to Theorem 2.7, we conclude the following result.

Proposition 3.2. Let Γ be an n-person cooperative game and S be a p.s.c. such that $\Pi(S) \neq \emptyset$ and assume that $S(P)$ is a unique point in E^n for each $P \in \Pi(S)$. Then $J_1(S) \neq \emptyset$.

In another approach, we model just the set of all viable coalition structures $\Pi(S)$ as the outcomes of an abstract game. A domination

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[†] In this section, Γ denotes an n-person cooperative game with side payments, without side payments or a game in partition function form.

relation on $\Pi(S)$ is defined as follows.

Definition 3.5. Let $P_1, P_2 \in \Pi(S)$, $\emptyset \neq R \in 2^N$ and S be a p.s.c. Then P_1 dominates P_2 via R w.r.t. p.s.c. S, denoted by $P_1 \operatorname{dom}_R(S) P_2$, iff

(3.4) $$R \in P_1 \text{ and}$$

(3.5) for each $y \in S(P_2)$, $\exists$ an $x \in S(P_1)$ such that $x_i > y_i \ \forall\ i \in R$.

Intuitively, if $P_1 \operatorname{dom}_R(S) P_2$, then the players in subset R prefer P_1 to P_2 because by Condition (3.5), no matter how the players disburse the payoffs corresponding to c.s. P_2, each player in R will do better in c.s. P_1. Condition (3.4) is imposed for the same reasons Condition (3.3) is imposed in Definition 3.2.

Definition 3.6. Let $P_1, P_2 \in \Pi(S)$ and S be a p.s.c. P_1 dominates P_2 w.r.t. S, denoted by $P_1 \operatorname{dom}(S) P_2$, iff

(3.6) $\exists$ a nonempty $R \in 2^N$ such that $P_1 \operatorname{dom}_R(S) P_2$.

We now have another abstract game $(\Pi(S), \operatorname{dom}(S))$ where $\Pi(S)$ is the set of outcomes and $\operatorname{dom}(S)$ is the binary relation on $\Pi(S)$. Once again we look at the core and the dynamic solution of this abstract game.

Definition 3.7. Let Γ be an n-person cooperative game and S be a p.s.c. The core of coalition structures w.r.t. p.s.c. S, denoted by $K_0(S)$, is the core of the abstract game $(\Pi(S), \operatorname{dom}(S))$.

Definition 3.8. Let Γ be an n-person cooperative game and S be a p.s.c. The dynamic solution of coalition structures w.r.t. p.s.c. S, denoted by $K_1(S)$, is the dynamic solution of the abstract game $(\Pi(S), \mathrm{dom}(S))$.

Once again, by appealing to Proposition 2.5, we have:

Proposition 3.3. $K_0(S) \subset K_1(S)$.

Also, since $\Pi(S)$ is always finite, we have:

Proposition 3.4. $K_1(S) \neq \emptyset$.

The following results gives a comparison of the two models.

Theorem 3.5. Let Γ be an n-person cooperative game and S be a p.s.c. Then we have

$$K_0(S) \supset \{P \in \Pi: (x,P) \in J_0(S)\}.$$

Proof: Let $P_1 \in \{P \in \Pi: (x,P) \in J_0(S)\}$. Then $\exists\, x \in S(P_1)$ such that (x,P_1) is undominated in $SC(S)$ which implies that P_1 is undominated (w.r.t. S) in $\Pi(S)$, i.e., $P_1 \in K_0(S)$. $\square$

Another consequence of the definitions of $K_0(S)$ and $J_0(S)$ is as follows:

Theorem 3.6. Let Γ be an n-person cooperative game and S be a p.s.c. such that $\forall\, P \in \Pi$, $S(P)$ is either a single point set in E^n or an empty set. Then

$$K_0(S) = \{P \in \Pi: (x,P) \in J_0(S)\} \quad \text{and}$$

$$J_0(S) = \{(S(P),P): P \in K_0(S)\}.$$

If $J_0(S) \neq \emptyset$, then the solution configuration model indicates both coalition structures and distribution of payoffs among the players as solutions in $J_0(S)$ whereas the coalition structure model indicates only coalition structures as solutions in $K_0(S)$. Also by Theorem 3.5, $J_0(S)$ indicates fewer (or at most an equal number of) coalition structures as solutions compared to $K_0(S)$. However, if the p.s.c. S is such that for each $P \in \Pi$, $S(P)$ is either a single point in E^n or an empty set, then the two models are identical (except in form) and indicate the same results.

3.3 Representation by Digraphs

Since the number of coalition structures is finite, we can represent the abstract game $(\Pi(S),\mathrm{dom}(S))$ of a game on N by means of a directed graph (or digraph). Given a payoff solution concept S, let $D = D(S)$ be a digraph whose vertex set $V(D) = \Pi(S)$ and whose arc set $A(D)$ is given by

$$A(D) = \{(P_1,P_2) \in \Pi(S) \times \Pi(S): P_1 \ \mathrm{dom}(S)\ P_2\}. \tag{3.7}$$

We call such a digraph D the domination digraph of the abstract game $(\Pi(S),\mathrm{dom}(S))$.

Example 3.1. Let Γ be a 3-person game on $\{1,2,3\}$. Let S be a p.s.c. defined as follows:

Let $0 \leq a \leq b \leq c = d$ be real numbers such that $c > a+b$ and

$$S(P) = \begin{cases} (0,0,0) & \text{if } P = \{\{1\}, \{2\}, \{3\}\} \\ (0,a,0) & \text{if } P = \{\{1,2\}, \{3\}\} \\ (0,0,b) & \text{if } P = \{\{1,3\}, \{2\}\} \\ \{(0, x_2, c-x_2): a \leq x_2 \leq c-b\} & \text{if } P = \{\{1\}, \{2,3\}\} \text{ or } \{\{1,2,3\}\} \end{cases}$$

To condense notation, we shall drop the braces around coalitions in coalition structures and, for example, denote $\{\{1\}, \{2,3\}\}$ by (1)(23). Note that

$$(1)(23) \ \mathrm{dom}(S) \ (1)(2)(3),$$

$$(1)(23) \ \mathrm{dom}(S) \ (12)(3),$$

$$(1)(23) \ \mathrm{dom}(S) \ (13)(2).$$

The domination graph of the game Γ is shown in Figure 3.1.

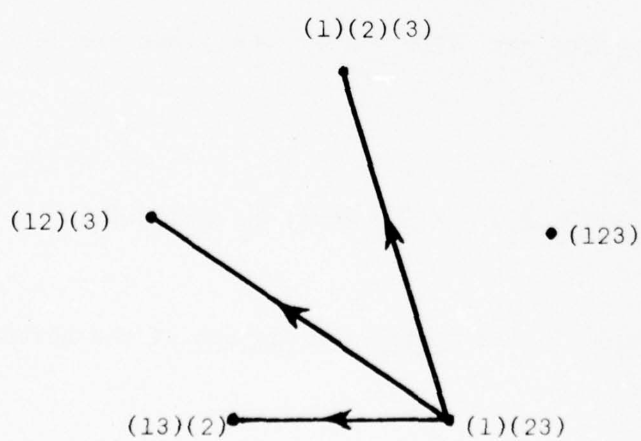


Figure 3.1. The domination digraph of game in Example 3.1.

Let $(P_1,P_2) \in A(D)$. Then we say P_1 is adjacent to P_2 and P_2 is adjacent from P_1. The outdegree, $od(P)$, for $P \in \Pi(S)$ is the number of c.s.'s adjacent from it and the indegree, $id(P)$, for $P \in \Pi(S)$ is the number adjacent to it. Then, in terms of this terminology, the core of the abstract game $(\Pi(S), dom(S))$ is given by

$$K_0(S) = \{P \in V(D): id(P) = 0\}. \tag{3.8}$$

In Example 3.1, $K_0(S) = \{(1)(23), (123)\}$.

The converse digraph D' of D has the same vertex set as D and the arc $(P_1,P_2) \in A(D') \Longleftrightarrow (P_2,P_1) \in A(D)$. Thus the converse of D is obtained by reversing the direction of every arc in D. If $D = D(S)$ is the domination digraph of the abstract game $(\Pi(S), dom(S))$, then we call its converse $D' = D'(S)$ the transition digraph of the abstract game $(\Pi(S), dom(S))$. The transition digraph of the game in Example 3.1 is shown in Figure 3.2.

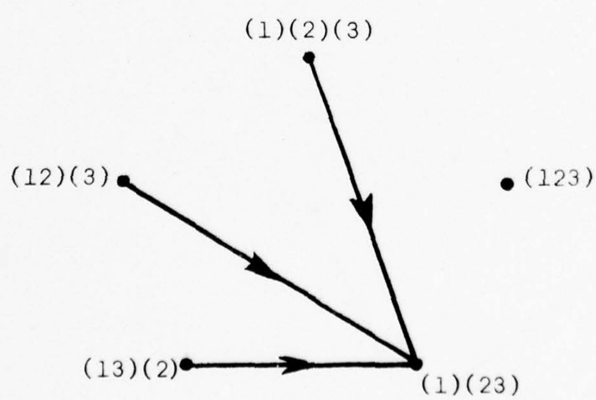
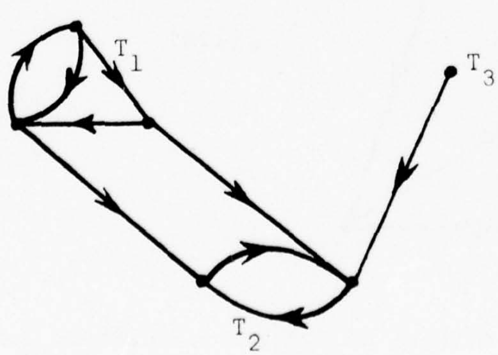


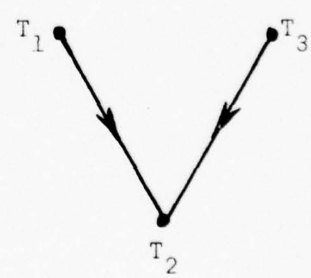
Figure 3.2. The transition digraph of the game in Example 3.1

To define the dynamic solution in terms of the transition graph, we need a few basic definitions from graph theory (cf. Harary [37]). A <u>(directed) walk</u> in a digraph is an alternating sequence of vertices and arcs $P_0, e_1, P_1, \ldots, e_n, P_n$ in which each arc e_i is (P_{i-1}, P_i). A <u>closed walk</u> has the same first and last vertex. A <u>path</u> is a walk in which all vertices are distinct; a <u>cycle</u> is a nontrivial closed walk with all vertices distinct (except the first and the last). If there is a path from P_1 to P_2, then P_2 is said to be <u>accessible from</u> P_1. A digraph is <u>strongly connected</u> or <u>strong</u> if any two vertices are mutually accessible. A <u>strong component</u> of a digraph is a maximal strong subgraph. Let $T_1, T_2, \ldots, T_m$ be the strong components of D'. The <u>condensation</u> D* of D has the strong components of D as its vertices, with an arc from T_i to T_j whenever there is at least one arc in D from a vertex of T_i to a vertex of T_j. (See Figure 3.3.) It follows from the maximality of strong components that the condensation D* of any graph D has no cycles. Let $D'(S)$ be the transition graph of the abstract game $(\Pi(S), \mathrm{dom}(S))$ with strong components $T_1, T_2, \ldots, T_m$.

D :



D*:



<u>Figure 3.3.</u> A digraph and its condensation.

Then the dynamic solution of the abstract game is given by

$$(3.9) \qquad K_1(S) = \cup\{T_i : od(T_i) = 0 \text{ in the condensation } D'^*\}.$$

In Example 3.1, $K_1(S) = \{(1)(23), (123)\}$.

3.4 Solutions with Respect to the Individually Rational Payoffs

Let (N,v) be an n-person cooperative game with side payments as defined in Section 2.5, Chapter II. The individually rational payoffs corresponding to coalition structure $P = (P_1,\ldots,P_m) \in \Pi$ is the set

$$I(P) = \{x \in E^n : \sum_{i \in P_j} x_i = v(P_j) \text{ for all } j = 1,\ldots,m \text{ and } x_i \geq v(i) \text{ for all } i \in N\}.$$

When $P = (N)$, $I((N))$ is also referred to as the set of imputations. Since $I(P)$ is nonempty for all $P \in \Pi$, we have

$$\Pi(I) = \Pi.$$

A game (N,v) is said to be superadditive if

$$(3.10) \qquad R_1 \cap R_2 = \emptyset,\ R_1,R_2 \in 2^N \Rightarrow v(R_1) + v(R_2) \leq v(R_1 \cup R_2)$$

and strictly superadditive if strict inequality holds in Relation (3.10).

Define the worth of a coalition structure P in the game (N,v) by

(3.11) $$w(\mathcal{P}) = \sum_{P_j \in \mathcal{P}} v(P_j).$$

Let

$$z = \max_{\mathcal{P} \in \Pi} w(\mathcal{P})$$

and define

(3.12) $$\Pi_z = \{\mathcal{P} \in \Pi : w(\mathcal{P}) = z\}.$$

If $x \in E^n$ and $R \subset N$, let $x(R)$ denote $\sum_{i \in R} x_i$. Then we have the following theorem.

Theorem 3.7. Let Γ be an n-person cooperative game with side payments. Then $K_0(I) \neq \emptyset$. In particular, we have $K_0(I) \supset \Pi_z$.

Proof. Let $\mathcal{P}^1 = (P_1^1, P_2^1, \ldots, P_m^1) \in \Pi_z$. Suppose $\exists\ \mathcal{P} \in \Pi$ such that $\mathcal{P}$ dom(I) $\mathcal{P}^1$, i.e. $\exists\ R \in \mathcal{P}$ such that $\mathcal{P}$ $\text{dom}_R(I)$ $\mathcal{P}^1$. Now we can write $R = \bigcup_{i=1}^{m} (R \cap P_i^1)$. Pick $y \in I(\mathcal{P}^1)$ such that $y(R \cap P_i^1) = v(P_i^1)$ if $R \cap P_i^1 \neq \emptyset$ for all $i = 1,2,\ldots,m$. Since $\mathcal{P}$ $\text{dom}_R(I)$ $\mathcal{P}^1$, $\exists\ x \in I(\mathcal{P})$ s.t. $x_i > y_i$ for all $i \in R$. I.e. $v(R) = x(R) > y(R) = \sum_{P_i^1 \cap R \neq \emptyset} v(P_i^1)$. Pick $\mathcal{P}^2 \in \Pi$ as follows. $\mathcal{P}^2 = \{R\} \cup \{\mathcal{P}^1 - \{P_i^1 : P_i^1 \cap R \neq \emptyset\}\}$ $\cup \{P_i^1 - R : P_i^1 \cap R \neq \emptyset\}$. Then $w(\mathcal{P}^2) > w(\mathcal{P}^1)$, a contradiction! This completes the proof. $\square$

The following example will show that, in general, we cannot make a stronger statement than in the theorem above.

Example 3.2. Let Γ be a 4-person game with

$$v(12) = v(34) = v(23) = 1, \text{ and } v(R) = 0 \text{ for all other } R \subset N.$$

Let $P_1 = (12)(34)$, $P_2 = (14)(23)$ and $P_3 = (1)(23)(4)$. $w(P_1) = 2$, $w(P_2) = w(P_3) = 1$. But $K_0(I) = \{P_1, P_2, P_3\}$.

However, with a slight assumption, we can claim the following.

Theorem 3.8. Let Γ be an n-person game with side payments such that $(N) \in \Pi_z$. Then $K_0(I) = \Pi_z$.

Proof: From Theorem 3.7 we need prove only $K_0(I) \subset \Pi_z$. Let $P_1 \in \Pi$ such that $P_1 \notin \Pi_z$, i.e. $w(P_1) < z$. Then (N) dom(I) P_1. This is seen as follows. Let $x \in I(P_1)$. Then $x(N) = w(P_1) < z$. Define y so that $y_i = x_i + (z - w(P_1))/n$ for all $i \in N$. Then $y \in I(\{N\})$ and $y_i > x_i$ for all $i \in N$. $\square$

Corollary 3.9. Let Γ be a superadditive game. Then $K_0(I) = \Pi_z$. Furthermore, if Γ is strictly superadditive, then $K_0(I) = \{(N)\}$.

Proof: Γ superadditive $\Rightarrow (N) \in \Pi_z$, and Γ strictly superadditive $\Rightarrow \Pi_z = \{(N)\}$. $\square$

For the solution configurations model, no general existence result is possible as is illustrated by the following example:

Example 3.3. Let $\Gamma = (N,v)$ be a 3-person game with

$$v(12) = v(13) = v(23) = 2, \quad v(123) = 2.5.$$

It can easily be shown that for this game $J_0(I) = \emptyset$.

3.5 Solutions with Respect to the Core

Let (N,v) be a cooperative game with side payments. Then the core of the game (N,v) corresponding to c.s. $P \in \Pi$ is defined by

$$(3.13) \qquad Co(P) = \{x \in I(P) : x(R) \geq v(R) \text{ for } \underline{\text{all}} \ R \in 2^N\}.$$

The core corresponding to a particular c.s. may be empty. Hence in general $\Pi(Co) \neq \Pi$. In fact, for some games the core corresponding to every c.s. may be empty, i.e., $\Pi(Co) = \emptyset$. A characterization of $K_0(Co)$ and $J_0(Co)$ is as follows.

Theorem 3.10. Let (N,v) be a cooperative game with side payments. Then,

$$K_0(Co) = \Pi(Co) = \{P : Co(P) \neq \emptyset\}.$$

Also

$$J_0(Co) = SC(Co) = \bigcup_{P \in \Pi(Co)} [Co(P) \times \{P\}].$$

Proof: Let $P_1, P_2 \in \Pi(Co)$. Suppose $P_1 \operatorname{dom}_R(Co) P_2$ for some $R \in P_1$. Let $y \in Co(P_2)$. Then $\exists\, x \in Co(P_1)$ s.t. $x_i > y_i$ for all $i \in R$. I.e. $x(R) > y(R)$. But since $R \in P_1$, $x(R) = v(R)$. Hence $y(R) < v(R)$ contradicting the fact that $y \in Co(P_2)$. The proof of the second assertion is similar to the first. $\square$

Corollary 3.11. Let (N,v) be a cooperative game with side payments. Let S be a p.s.c. such that, for all $P \in \Pi$, $S(P) \subset I(P)$, and

$S(P) \cap Co(P) \neq \emptyset$ whenever $Co(P) \neq \emptyset$. Then $K_0(Co) \subset K_0(S)$ and $J_0(Co) \subset J_0(S)$ (as subsets of Π).

In light of Theorem 3.10 we would like to characterize the coalition structures with nonempty cores. The next two theorems along with a known characterization of games with nonempty cores corresponding to the grand coalition N accomplish this task.

Theorem 3.12. Let (N,v) be a cooperative game with side payments. If $\Pi(Co) \neq \emptyset$, then $\Pi(Co) = \Pi_z$.

Proof: Let $P_1 \in \Pi(Co)$, and suppose $P_1 \notin \Pi_z$. Then $\exists\ P_2 \in \Pi$ such that $w(P_2) > w(P_1)$. Let $x \in Co(P_1)$. Then $x(R) \geq v(R)$ for all $R \subset N$ which implies that $w(P_1) = x(N) \geq w(P_2)$ and this is a contradiction! Hence $\Pi(Co) \subset \Pi_z$.

Let $P_1 \in \Pi_z$ and assume $P_2 \in \Pi(Co) \subset \Pi_z$. Let $x \in Co(P_2)$. Then $x(R) \geq v(R)$ for all $R \subset N$. If $x(P) > v(P)$ for some $R \in P_1$, then $w(P_2) = x(N) > w(P_1)$, contradicting the fact that $P_1 \in \Pi_z$. Hence $x(P) = v(P)$ for all $P \in P_1 \Rightarrow x \in Co(P_1) \Rightarrow P_1 \in \Pi(Co)$. Therefore $\Pi(Co) \supset \Pi_z$. $\square$

Corollary 3.13. Let (N,v) be a game with side payments. Then for all $P_1, P_2 \in \Pi(Co)$, $Co(P_1) = Co(P_2)$.

Corollary 3.14. Let (N,v) be a game with side payments. If there is a $P \in \Pi_z$ such that $Co(P) = \emptyset$, then $\Pi(Co) = \emptyset$.

Given a game $\Gamma = (N,v)$ define a game $\Gamma_z = (N,v_z)$ derived from Γ as follows.

$$(3.14) \qquad v_z(R) = \begin{cases} z & \text{if } R = N \\ v(R) & \text{for all other } R \subset N \end{cases}$$

$$\text{where} \quad z = \max_{P \in \Pi} w(P).$$

When there is more than one game under discussion, we shall denote the sets $Co(P)$, $\Pi(Co)$ and Π_z by $Co(P,\Gamma)$, $\Pi(Co,\Gamma)$, and $\Pi_z(\Gamma)$, respectively.

Theorem 3.15. Let $\Gamma = (N,v)$ be a game and Γ_z be as in Relation (3.14). Then if $Co(P,\Gamma) \neq \emptyset$, $Co(P,\Gamma) = Co((N),\Gamma_z)$.

Proof: From the definition of Γ_z it is clear that for $P \neq (N)$ $Co(P,\Gamma) = Co(P,\Gamma_z)$. From Theorem 3.12 we obtain $\Pi(Co,\Gamma_z) = \Pi_z(\Gamma_z)$. Since $(N) \in \Pi_z(\Gamma_z)$, by Corollary 3.13, $Co(P,\Gamma_z) = Co((N),\Gamma_z)$. Hence the theorem follows. $\square$

Games with nonempty cores corresponding to the grand coalition have been characterized by Bondareva [19,20] and Shapley [93]. For the sake of completeness we will repeat this characterization here.

A balanced set $\mathcal{B}$ is defined to be a collection of subsets R of N with the property that there exist positive numbers δ_R $\forall R \in \mathcal{B}$ called weights, such that for each $i \in N$ we have

$$(3.15) \qquad \sum_{\{R \in \mathcal{B}:\ i \in R\}} \delta_R = 1.$$

A game (N,v) is called balanced iff

(3.16)
$$\sum_{R \in \mathcal{B}} \delta_R v(R) \leq v(N)$$

holds for every balanced set with weights $\{\delta_R\}$. The following theorem was proved by Bondareva [19,20] and Shapley [93].

Theorem 3.16. Let (N,v) be a game. Then $Co((N)) \neq \emptyset$ if and only if the game is balanced.

Corollary 3.17. Let $\Gamma = (N,v)$ be a game. Then $\Pi(Co,\Gamma) \neq \emptyset$ if and only if the game (N,v_z) is balanced.

Proof: (Necessity): $\Pi(Co,\Gamma) \neq \emptyset \Rightarrow Co((N),(N,v_z)) \neq \emptyset$ (by Theorem 3.15) $\Rightarrow (N,v_z)$ is balanced (by Theorem 3.16).
(Sufficiency): If $\Gamma_z = (N,v_z)$ is balanced $\Rightarrow Co((N),\Gamma_z) \neq \emptyset$ (by Theorem 3.16). If $(N) \in \Pi_z(\Gamma)$ then $\Gamma = \Gamma_z$ and we are finished. Otherwise $\exists\ P \in \Pi_z(\Gamma_z)$ such that $P \neq (N)$. Then, $Co(P,\Gamma) = Co(P,\Gamma_z) = Co((N),\Gamma_z) \neq \emptyset$. $\square$

Thus we have completely characterized $K_0(Co)$ and $J_0(Co)$ for all games with side payments.

Example 3.4. (A game with no solution. See Lucas [58,59].)
Let $N = \{1,2,3,4,5,6,7,8,9,10\}$ and v be given by

$$v(N) = 5, \quad v(13579) = 4,$$
$$v(12) = v(34) = v(56) = v(78) = v(9\,10) = 1,$$
$$v(3579) = v(1579) = v(1379) = 3,$$
$$v(357) = v(157) = v(137) = 2,$$

$$v(359) = v(159) = v(139) = 2,$$

$$v(1479) = v(3679) = v(5279) = 2, \quad \text{and}$$

$$v(R) = 0 \quad \text{for all other} \quad R \subset N.$$

In this game $z = 5$, $\Pi_z = \{(N), P_1 = (12)(34)(56)(78)(9\,10)\}$ and $Co((N)) = Co(P_1) = \{x\colon x(12) = x(34) = x(56) = x(78) = x(9\,10) = 1,$ and $x(13579) \geq 4\}$. By Theorem 3.10,

$$K_0(Co) = \{(N), P_1\}, \quad \text{and}$$

$$J_0(Co) = Co((N)) \times \{(N), P_1\}.$$

3.6 Solutions with Respect to the Shapley Value

Shapley [89] defined a unique value satisfying three axioms for all n-person cooperative games with side payments. It was assumed that the grand coalition would form. Later, Aumann and Dreze [8] generalized the axioms to define the Shapley value for all coalition structures.

A permutation α of N is a one-one function from N onto itself. For $R \in 2^N$, write $\alpha R = \{\alpha i\colon i \in R\}$. If v is a game on N, define a game $\alpha * v$ on N by

$$(3.17) \qquad (\alpha * v)(R) = v(\alpha R) \quad \text{for all} \quad R \in 2^N.$$

Also, if v and u are games on N, define a game $v+u$ on N by

$$(3.18) \qquad (v+u)(R) = v(R) + u(R) \quad \text{for all} \quad R \in 2^N.$$

Call a c.s. $\mathcal{P} = (P_1,\ldots,P_m)$ invariant under α if $\alpha P_j = P_j$ for all $j = 1,\ldots,m$. Player i is null if $v(R \cup (i)) = v(R)$ for all $R \in 2^N$. Let G^N denote the set of all games with side payments on N. Since we assume that for all games with side payments, $v(\emptyset) = 0$ and $v(i) = 0$ $\forall\ i \in N$, G^N is a Euclidean space of dimension $2^n - (n+1)$.

Fix $N = \{1,\ldots,n\}$ and $\mathcal{P} = (P_1,\ldots,P_m) \in \Pi$. The Shapley value corresponding to c.s. $\mathcal{P}$ is a function $\Phi_{\mathcal{P}}$ from G^N to E^n i.e. a function that associates with each game a payoff vector satisfying the following axioms:

A.1 (Relative Efficiency): $\Phi_{\mathcal{P}}(v)(P_j) = v(P_j)$ for all $j = 1,\ldots,m$.

A.2 (Symmetry): For all permutations α of N under which $\mathcal{P}$ is invariant,

$$\Phi_{\mathcal{P}}(\alpha * v)(R) = \Phi_{\mathcal{P}}(v)(\alpha R).$$

A.3 (Additivity): $\Phi_{\mathcal{P}}(v+u) = \Phi_{\mathcal{P}}(v) + \Phi_{\mathcal{P}}(u)$.

A.4 (Null Player Axiom): If i is a null player, then $\Phi_{\mathcal{P}}(v)(i) = 0$.

When $\mathcal{P} = (N)$, the above axioms are equivalent to Shapley's axiom which specify a unique value $\phi(v) = (\phi_1(v),\ldots,\phi_n(v))$ given by

$$(3.19)\quad \phi_i(v) = \Phi_{(N)}(v)(i) = \sum_{R\subset N} \frac{(r-1)!(n-r)!}{n!}\,[v(R) - v(R - \{i\})]$$

where $r = |R|$, the cardinality of coalition R. For each $R \in 2^N$, denote by $v|R$ the game on R defined for all $T \subset R$ by

(3.20) $$(v|R)(T) = v(T).$$

Theorem 3.18. Fix N and $\mathcal{P} = (P_1,\dots,P_m)$. Then there is a unique value $\Phi_{\mathcal{P}}$ and it is given for all $j = 1,\dots,m$ and $i \in P_j$ by

(3.21) $$(\Phi_{\mathcal{P}}v)(i) = (\Phi_{(P_j)}(v|P_j))(i).$$

Proof: See Aumann and Dreze [8, pp. 220-221].

Since $\Phi(\mathcal{P})$[†] is nonempty for all $\mathcal{P} \in \Pi$, $\Pi(\Phi) = \Pi$. Also note from (3.19) that if v is superadditive, then $\Phi(\mathcal{P})(i) \geq 0$, hence $\Phi(\mathcal{P}) \in I(\mathcal{P})$. Also, since $\Phi(\mathcal{P})$ consists of a unique outcome for all $\mathcal{P} \in \Pi$, by Theorem 3.6 the s.c. model and the c.s. model give identical results. For convenience, all the results in this section are stated only for the c.s. model.

A partial existence theorem for $K_0(\Phi)$ is as follows:

Theorem 3.19. Let Γ be an n-person game in which the only coalitions with positive values are all the (n-1)-person and n-person coalitions. Then $K_0(\Phi) \neq \emptyset$.

Proof: Let us denote the game as follows:

$$v(i) = 0 \quad \text{for all} \quad i \in N,$$
$$v(N - (i)) = a_i \quad \text{for all} \quad i \in N,$$
$$v(N) = b, \quad \text{and} \quad v(R) = 0 \quad \text{for all other} \quad R \subset N.$$

[†]When there is no doubt about the game v under consideration, we shall denote $\Phi_{\mathcal{P}}(v)$ by $\Phi(\mathcal{P})$ which is consistent with the previous section.

We can assume (by relabelling of the players) that

$$a_1 \leq a_2 \leq \dots \leq a_n. \tag{3.22}$$

Let $a = \sum_{i=1}^{n} a_i$ and $\Pi_{a_n} = \{P \in \Pi: w(P) = a_n\}$. Using (3.19) and (3.21) we have

$$\Phi((N))(i) = ((n-1)b + a - n\cdot a_i)/(n(n-1)). \tag{3.23}$$

By (3.22) we have

$$\Phi((N))(1) \geq \Phi((N))(2) \geq \dots \geq \Phi((N))(n) \tag{3.24}$$

Also,

$$\Phi((N-i)(i))(j) = \begin{cases} a_i/(n-1) & \text{for } j = 1,\dots,n \\ & \phantom{\text{for }} j \neq i \\ 0 & \text{for } j = i \end{cases} \tag{3.25}$$

Clearly, the only c.s.'s we need look at are (N) and $(N-i)(i)$ for $i = 1,\dots,n$. All the c.s.'s not in Π_{a_n} (except (N)) are dominated by c.s.'s in Π_{a_n}. From Expressions (3.23), (3.24) and (3.25) it follows that (N) dom(Φ) $(N-n)(n)$ iff

$$\Phi((N))(n-1) > \Phi((N-n)(n))(n-1)$$

i.e. iff

$$b > (n(a_n + a_{n-1}) - a)/(n-1).$$

Also if $a_n = a_{n-1}$ (i.e. $(N - (n-1))(n-1) \in \Pi_{a_n}$) then

$$(N) \text{ dom}(\Phi) \ (N - (n-1))(n-1)$$

iff

$$\Phi((N))(n) > \Phi((N - (n-1))(n-1))(n),$$

i.e. iff

$$b > (n(a_n + a_{n-1}) - a)/(n-1).$$

Now,

$$(N-n)(n) \text{ dom}(\Phi) \ (N)$$

iff

$$\Phi((N-n)(n))(1) > \Phi((N))(1),$$

i.e. iff

$$b < (n(a_n + a_1) - a)/(n-1).$$

Hence we have

$$K_0(\Phi) = \begin{cases} (N) & \text{If } b > (n(a_n + a_{n-1}) - a)/(n-1) \\ \Pi_{a_n} & \text{if } b < (n(a_n + a_1) - a)/(n-1) \\ (N) \cup \Pi_{a_n} & \text{otherwise.} \end{cases}$$

□

Corollary 3.20. Let Γ be a 3-person game with side payments. Then $K_0(\Phi) \neq \emptyset$.

In general, this is the strongest existence result we can obtain. I.e. there is a 4-person game for which $K_0(\Phi) = \emptyset$. This is shown in Example 3.9.

If $Co(P) \neq \emptyset$, $\Phi(P)$ may not belong to $Co(P)$. Hence Corollary 3.11 is not applicable for the Shapley value. The following example illustrates this fact.

Example 3.5. Let $N = \{1,2,3\}$ and v be given by $v(1) = v(2) = v(3) = 0$, $v(12) = 50$, $v(13) = 50$, $v(23) = 56$, and $v(123) = 80$. Then the Shapley value is given by:

$$\Phi(P) = \begin{cases} (24.67,\ 27.67,\ 27.67) & \text{if } P = (123) \\ (0,\ 28,\ 28) & \text{if } P = (1)(23) \\ (25,\ 0,\ 25) & \text{if } P = (13)(2) \\ (25,\ 25,\ 0) & \text{if } P = (12)(3) \\ (0,\ 0,\ 0) & \text{if } P = (1)(2)(3) \end{cases}$$

Note that $Co((123)) = Conv\{(20, 30, 30), (24, 26, 30), (24, 30, 26)\}$ but $\Phi((123)) \notin Co((123))$. The transition digraph is shown in Figure 3.4, and hence $K_0(\Phi) = K_1(\Phi) = (1)(23)$.

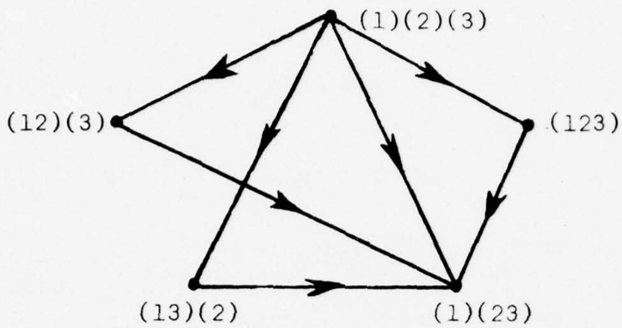


Figure 3.4. The transition digraph for Example 3.5.

The above example illustrates a weakness of the Shapley value in that the Shapley value is derived entirely from the characteristic function rather than the bargaining positions of the players in the process of coalition formation. However, the Shapley value has been extensively used as an a priori measure of power of players in "simple" games. Hence the study of $K_0(\Phi)$ and $K_1(\Phi)$ is most appropriate for simple games.

The class of all simple games forms a subclass of the class of all cooperative games with side payments. A simple game is a game in which every coalition has value either 1 or 0. A coalition $R \subset N$ is winning if $v(R) = 1$ and losing if $v(R) = 0$. A simple game can be represented by a pair $(N,\mathcal{W})$ where N is the set of players and $\mathcal{W}$ is the set of winning coalitions. A simple game is monotonic iff $R \in \mathcal{W}$ and $T \supset R \Rightarrow T \in \mathcal{W}$, and superadditive (or proper) iff $R \in \mathcal{W} \Rightarrow N - R \notin \mathcal{W}$. Superadditivity implies monotonicity in simple games. A winning coalition R is called minimal winning if every proper subset of R is losing. A monotonic simple game can be represented by the pair $(N,\mathcal{W}^m)$ where $\mathcal{W}^m$ is the set of all minimal winning coalitions. If $\mathcal{W}^m = \{\{i\}\}$, then player i is said to be a dictator. If $j \in \cap\mathcal{W}^m \neq \emptyset$, then player j

is said to be a veto player. If $k \notin \cup \mathcal{W}^m$ then player k is said to be a dummy. Dummies play no active role in the game and for all practical purposes can be omitted from the set of players. A weighted majority game is a monotonic simple game that can be represented by

$$(3.26) \qquad [q: a_1, a_2, \ldots, a_n]$$

where $q \geq 0$ is called the quota, $a_i \geq 0$, $i = 1,\ldots,n$ is the weight of the i^{th} player, and $R \in \mathcal{W} \iff \sum_{i \in R} a_i \geq q$. Expression (3.26) is said to be a weighted majority representation of the simple game. Two weighted majority representations are said to be equivalent if they represent the same simple game. E.g. $[2; 1,1,1]$ and $[5; 2,3,4]$ are equivalent since both represent the game $(\{1,2,3\}, \mathcal{W}^m = \{(12), (13), (23)\})$. Not every monotonic simple game may have a weighted majority representation.

Example 3.6. The most common of all simple games is the straight majority game M_n, n odd, in which

$$\mathcal{W}^m = \{R \subset N: |R| = (n+1)/2\}$$

where $|R|$ denotes the cardinality of coalition R. The Shapley value is given by

$$\Phi(P)(i) = \begin{cases} 1/|R| & \text{if } i \in R \in \mathcal{W},\ R \in P \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that

$K_1(\Phi) = K_0(\Phi) = \{P \in \Pi\colon\ P$ contains a minimal winning coalition$\}$.

Example 3.7. The pure bargaining game B_n, is given by $\mathcal{W}^m = \{(N)\}$. The Shapley value is given by

$$\Phi(P)(i) = \begin{cases} 1/n & \text{if } P = (N) \\ 0 & \text{otherwise} \end{cases}$$

clearly, $K_1(\Phi) = K_0(\Phi) = \{(N)\}$.

Example 3.8. Let Γ be a proper game with a dictator. Then

$$\Phi(P)(i) = \begin{cases} 1 & \text{if } i \text{ is a dictator} \\ 0 & \text{otherwise.} \end{cases}$$

Hence $K_1(\Phi) = K_0(\Phi) = \Pi$. Note that every player who is not a dictator is a dummy. So essentially we have a 1-person game in which the only player is winning by himself.

Example 3.9. Consider the weighted majority game $[3;\ 2,1,1,1]$. The minimal winning coalitions are $\mathcal{W}^m = \{(12), (13), (14), (234)\}$. The Shapley value is given by

$$\phi(P) = \begin{cases} (1/2,\ 1/6,\ 1/6,\ 1/6) & \text{if } P = (1234) \\ (2/3,\ 1/6,\ 1/6,\ 0) & \text{if } P = (123)(4) \\ (2/3,\ 1/6,\ 0,\ 1/6) & \text{if } P = (124)(3) \\ (2/3,\ 0,\ 1/6,\ 1/6) & \text{if } P = (134)(2) \\ (1/2,\ 1/2,\ 0,\ 0) & \text{if } P = (12)(34) \text{ or } (12)(3)(4) \\ (1/2,\ 0,\ 1/2,\ 0) & \text{if } P = (13)(24) \text{ or } (13)(2)(4) \\ (1/2,\ 0,\ 0,\ 1/2) & \text{if } P = (14)(23) \text{ or } (14)(2)(3) \\ (0,\ 1/3,\ 1/3,\ 1/3) & \text{if } P = (1)(234) \\ (0,\ 0,\ 0,\ 0) & \text{otherwise.} \end{cases}$$

The transition digraph of the game is shown in Figure 3.6. Since all c.s.'s that contain only losing coalitions are dominated, these are omitted from this transition digraph. Note that $K_0(\Phi) = \emptyset$. However,

$$K_1(\Phi) = \{(1)(234),\ (12)(3)(4),\ (12)(34),\ (134)(2),\ (13)(24)$$
$$(13)(2)(4),\ (124)(3),\ (14)(23),\ (14)(2)(3),\ (123)(4)\}.$$

A closer look at the Shapley value for different c.s.'s in Example 3.9 reveals the following observation. If players 1 and 2 who are in a winning coalition with 3 in the c.s. (123)(4) decide to expel player 3 from the coalition and form the smaller winning coalition (12), one would expect both players not to decrease their power in the smaller winning coalition (12) since there are fewer players to share the same amount of power. However, player 1 actually does decrease his power from 2/3 to 1/2. We shall call this phenomenon the paradox of smaller coalitions. To understand why this phenomenon occurs, let us look at Theorem 3.18. It states

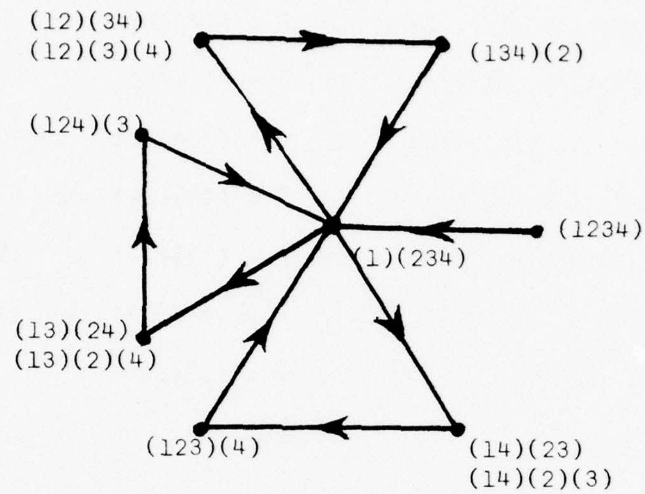


Figure 3.5. The transition digraph in Example 3.9.

that given a c.s. $P = (P_1,\ldots,P_m)$ the Shapley value of player i in coalition P_j depends only on the subgame $v|P_j$. I.e. the Shapley value of a player in a coalition is oblivious of the presence of other players not in the coalition for bargaining purposes. We shall regard this phenomenon as a "flaw" in the properties of the Shapley value. To make the above discussion more formal, let $\Gamma = (N,\mathcal{W})$ be a simple game and σ be a payoff value concept (i.e. for all games and for each $P \in \Pi$, $\sigma(P)$ is a single point in E^n, where n = the number of players). We say Γ does not exhibit the paradox of smaller coalitions w.r.t. payoff value concept σ iff the following holds:

Let $P_1, P_2 \in \Pi$ such that $P_k \in P_1$, $P_k \in \mathcal{W}$, $P_{kl} \subset P_k$ is such that $P_{kl} \in \mathcal{W}$, and $P_{kl} \in P_2$. Then $\sigma(P_2)(i) \geq \sigma(P_1)(i)$ for all $i \in P_{kl}$.

The following result is a consequence of the above definition.

Theorem 3.21. Let Γ be a proper simple game that does not exhibit the paradox of smaller coalitions w.r.t. Φ. Then $K_0(\Phi) \neq \emptyset$.

Proof: Let $T \in \mathcal{W}^m$ such that $|T| \leq |R|$ for all $R \in \mathcal{W}^m$. Let $P \in \Pi$ be such that $T \in P$. Then $\Phi(P)(i) = 1/|T|$ for all $i \in T$. Suppose $\exists\, P_1 \in \Pi$ such that $P_1 \text{ dom}_R(\Phi)\, P$ for some $R \in P_1$, i.e., $\Phi(P_1)(i) > \Phi(P)(i)$ for all $i \in R$. Let R' be any minimal winning coalition contained in R, i.e. $R' \subset R$ and $R' \in \mathcal{W}^m$. Let $P_2 \in \Pi$ be such that $R' \in P_2$. Then since Γ does not exhibit the paradox, $\Phi(P_2)(i) \geq \Phi(P_1)(i)$ for all $i \in R'$. Also

$$\Phi(P_2)(i) = \begin{cases} 1/|R'| & \text{if } i \in R' \\ 0 & \text{otherwise.} \end{cases}$$

Since Γ is proper, $R' \cap T \neq \emptyset$. Hence for all $i \in R' \cap T$, $1/|R'| = \Phi(P_2)(i) \geq \Phi(P_1)(i) > \Phi(P)(i) = 1/|T|$, which is a contradiction (since $|R'| \geq |T|$)! $\square$

Let

(3.27) $$t = \min_{R \in \mathcal{W}^m} |R|$$

and let

(3.28) $\Pi_t = \{P \in \Pi: P \text{ contains a winning coalition of size } t\}$.

Then we obtain the following.

Corollary 3.22. Let Γ be a proper simple game that does not exhibit the paradox of smaller coalitions w.r.t. Φ. Then $K_0(\Phi) \supset \Pi_t$.

That in general we cannot strengthen the above result is shown by the following example.

Example 3.10. Let Γ be a 4-person game represented by $[4; 2,2,1,1]$. The minimal winning coalitions are $\{(12), (134), (234)\}$. The Shapley value is given by

$$\Phi(P) = \begin{cases} (1/2,\ 1/2,\ 0,\ 0) & \text{if } P = (12)(34) \text{ or } (12)(3)(4) \\ (1/2,\ 1/2,\ 0,\ 0) & \text{if } P = (123)(4) \text{ or } (124)(3) \\ (1/3,\ 0,\ 1/3,\ 1/3) & \text{if } P = (134)(2) \\ (0,\ 1/3,\ 1/3,\ 1/3) & \text{if } P = (1)(234) \\ (1/3,\ 1/3,\ 1/6,\ 1/6) & \text{if } P = (1234) \end{cases}$$

Note that the game does not exhibit the paradox of smaller coalitions. Also $t = 2$, and $\Pi_t = \{(12)(3)(4), (12)(34)\}$. However, $K_0(\Phi) = \{(12)(3)(4), (12)(34), (123)(4), (124)(3)\}$. Observe that players 3 and 4 are dummies in the subgame on $\{1,2,3\}$ and $\{1,2,4\}$ respectively.

An interesting problem raised by Theorem 3.21 is to characterize the class of games that do not exhibit the paradox of smaller coalitions w.r.t. Φ. Let us look at symmetric games. A game (N,v) is called symmetric if the value of a coalition depends only on the size of the coalition. A symmetric monotonic simple game is of the type $M_{n,k} = (N,\mathcal{W})$

where $W = \{R \subset N: |R| \geq k\}$. The following proposition follows from the symmetry axiom of the Shapley value.

Proposition 3.23. Let Γ be a symmetric simple game. Then Γ does not exhibit the paradox of smaller coalitions w.r.t. Φ. In fact, $K_0(\Phi) = \Pi_t$.

Proof: The Shapley value is given by

$$\Phi(P)(i) = \begin{cases} 1/|R| & \text{if } i \in R \in P \text{ and } R \in W \\ 0 & \text{otherwise.} \end{cases}$$

Hence the result follows from Statement (3.21). $\square$

Since Example 3.10 does not exhibit the paradox and is not symmetric, Proposition 3.23 is not a complete characterization. A list of all proper simple games with four or fewer players is given in the appendix along with the Shapley value Φ corresponding to all coalition structures, $K_0(\Phi)$, and whether or not the game exhibits the paradox.

Another interesting problem is to determine, if possible, a power index that has all the desirable properties of the Shapley value but that does not exhibit the paradox of smaller coalitions.

The most critical axiom of the Aumann-Dreze generalization of the Shapley value is A.3.

A.3. $\Phi_P(v+u) = \Phi_P(v) + \Phi_P(u)$.

This axiom is acceptable if and only if we assume that the c.s. P is fixed

and that players in a coalition $P_k \in \mathcal{P}$ cannot bargain on the basis of the values of coalitions not contained in P_k. This assumption is not appropriate for our model where the players are bargaining for a coalition structure and no c.s. is fixed.

Another generalization of the Shapley value (which he defined only for the grand coalition) to the case of all coalition structures which is appropriate for monotonic simple games is as follows.

(i) The Shapley value corresponding to the grand coalition is used as an a priori measure of power of the players. This is suggested by Shapley and Shubik [94].

(ii) And within any coalition in a c.s., a player can expect to share in the payoff proportional to his power as defined in (i). This is suggested by Gamson [34].

Assumptions (i) and (ii) define a unique value for all monotonic simple games which we denote by Φ'. We can define Φ' by axioms as follows:

The (generalized) Shapley value Φ' is a function from $\Pi \times G^N$ to E^n, i.e., a function that associates with each game and a c.s. a payoff vector satisfying the following axioms:

<u>A'.1</u> (<u>Relative Efficiency</u>): $\Phi'(\mathcal{P},v)(P_k) = v(P_k)$ for all $P_k \in \mathcal{P}$, and all $\mathcal{P} \in \Pi$.

<u>A'.2</u> (<u>Symmetry</u>): For all $\mathcal{P} \in \Pi$, and all permutations α of N under which $\mathcal{P}$ is invariant,

$$\Phi'(\mathcal{P},\alpha * v)(R) = \Phi'(\mathcal{P},v)(\alpha R) \quad \text{for all } R \subset N.$$

A'.3 (Additivity): If v and u are games in G^N, then

$$\Phi'((N),v+u) = \Phi'((N),v) + \Phi'((N),u)$$

A'.4 (Null Player Axiom): If i is a null player, then

$$\Phi'(P,v)(i) = 0 \text{ for all } P \in \Pi.$$

A'.5 (Proportionality): For all $P \in \Pi$,

$$\Phi'(P,v)(i) \cdot \Phi'((N),v)(j) = \Phi'(P,v)(j) \cdot \Phi'((N),v)(i)$$

for all $i,j \in P_k \in P$.

When $P = (N)$, Axioms A'.1 - A'.4 are equivalent to Shapley's axioms which specify the unique value given by Expression (3.19). Denote $\Phi'((N),v)$ by $\phi(v) = (\phi_1(v),\ldots,\phi_n(v))$. (Since $\Phi'((N),v) = \Phi_{(N)}(v)$ our notation is consistent.) Next we obtain the following result.

Theorem 3.24. Fix $N = \{1,\ldots,n\}$ and let G^N denote the set of all monotonic games on N. Then there is a unique value satisfying Axioms A'.1 - A'.5 given by Expression (3.19) and

$$(3.29)\quad \Phi'(P,v)(j) = \begin{cases} \dfrac{\phi_j(v)}{\sum_{i \in P_k} \phi_i(v)} \cdot v(P_k) & \text{where } P_k \in P \text{ is such that } j \in P_k, \text{ if } j \text{ is not a null player} \\ 0 & \text{if } j \text{ is a null player} \end{cases}$$

Proof: It can be easily shown that Statements (3.19) and (3.29) satisfy Axioms A'.1 - A'.5. Uniqueness follows from Axioms A'.1 and A'.5. $\square$

Corollary 3.25. Let Γ be a monotonic simple game. Then Γ does not exhibit the paradox of smaller coalitions w.r.t. Φ'.

Proof: This follows from Expression (3.29).

In view of Corollary 3.22, we might be tempted to assert that $K_0(\Phi') \supset \Pi_t$. However, the following example shows that it is not true.

Example 3.11. Consider the weighted majority game given in Example 3.9, [3; 2,1,1,1]. Then Φ' is given by

$$\Phi'(P)^{\dagger} = \begin{cases} (3/6,\ 1/6,\ 1/6,\ 1/6) & \text{if } P = (1234) \\ (3/5,\ 1/5,\ 1/5,\ 0) & \text{if } P = (123)(4) \\ (3/4,\ 1/4,\ 0,\ 0) & \text{if } P = (12)(3)(4) \text{ or } (12)(34) \\ (0,\ 1/3,\ 1/3,\ 1/3) & \text{if } P = (1)(234). \end{cases}$$

For all other c.s.'s, $\Phi'(P)$ can be determined by the symmetry of players 2, 3, and 4. It is clear that $K_0(\Phi') = \{(1)(234)\}$. Note that in this example $t = 2$, hence $(1)(234) \notin \Pi_t$.

Let

$$(3.30) \qquad s = \min_{R \in W^m} \sum_{i \in R} \phi_i(v),$$

---

† When there is no doubt about the game v under consideration, we shall denote $\Phi'(P,v)$ by $\Phi'(P)$ which is consistent with the established notation.

and let

(3.31) $\Pi_s = \{P \in \Pi\colon\ P$ contains a coalition R such that $\sum_{i \in R} \phi_i(v) = s\}$.

Then we have the following important fact.

Theorem 3.26. Let Γ be a proper simple game. Then $K_0(\Phi') = \Pi_s$.

Proof: Denote $\Phi'((N))$ by $\phi = (\phi_1,\ldots,\phi_n)$. Let $P_1 \in \Pi_s$. Suppose $P_2 \in \Pi$ such that $P_2 \operatorname{dom}_R(\Phi') P_1$ for some $R \in P_2$ such that $R \in P$. Then $\Phi'(P_2)(i) > \Phi'(P_1)(i)$ for all $i \in R$. Let $T \in P_1$ be such that $T \in W^m$ and $\sum_{i \in T} \phi_i = s$. Since Γ is proper $R \cap T \neq \emptyset$. Pick $j \in R \cap T$. Then $\Phi'(P_1)(j) = \phi_j/s$. Since $j \in R$, $\Phi'(P_2)(j) = \phi_j/(\sum_{i \in R} \phi_i) > \phi_j/s$, i.e., $\sum_{i \in R} \phi_i < s$, *a contradiction! Hence* $K_0(\Phi') \supset \Pi_s$.

Let $P_1 \in \Pi_s$ and $P_2 \in \Pi$ be such that $P_2 \notin \Pi_s$. Then $P_1 \operatorname{dom}_T(\Phi') P_2$ where $T \in P_1$ such that $T \in W^m$ and $\sum_{i \in T} \phi_i = s$ because $\Phi(P_1)(i) = \phi_i/s$ for all $i \in T$ and $\Phi'(P_2)(i) < \phi_i/s$ for all $i \in T$. Hence $K_0(\Phi') \subset \Pi_s$. $\square$

3.7 Solutions with Respect to the Bargaining Set $M_1^{(i)}$

The bargaining set was first introduced by Aumann and Maschler [9]. They defined several types of bargaining sets. One of these, denoted by $M_1^{(i)}$, was shown to be nonempty for every c.s. by Peleg [81].

Let x^R denote a vector in E^r where $r = |R|$, whose elements are indexed by the players in R. Let $x \in I(P)$ and let i and j be two distinct players in coalition $P_k \in P$. An objection of i against j to $x \in I(P)$ is a vector y^R, where R is a coalition containing player i

but not j, whose coordinates y_ℓ satisfy $y_i > x_i$, $y_\ell \geq x_\ell \; \forall \, \ell \in R$ and $\sum_{\ell \in R} y_\ell = v(R)$. A counter-objection to this objection is a vector z^D, where D is a coalition containing player j but not i, whose coordinates z_ℓ satisfy $z_\ell \geq x_\ell$ for each $\ell \in D$, $z_\ell \geq y_\ell$ for each $\ell \in R \cap D$, and $\sum_{\ell \in D} z_\ell = v(D)$.

$x \in I(P)$ is stable if for each objection to x, there is a counter-objection. The bargaining set corresponding to the c.s. $P \in \Pi$, denoted by $M_1^{(i)}(P)$ is the set of all stable individually rational payoff $x \in I(P)$, i.e.,

$$M_1^{(i)}(P) = \{x \in I(P) \colon x \text{ is stable}\}. \tag{3.32}$$

Theorem 3.27. Let Γ be an n-person cooperative game with side payments. Then $M_1^{(i)}(P) \neq \emptyset$ for each $P \in \Pi$.

Proof. See Davis and Maschler [29] and Peleg [81].

As a result $\Pi(M_1^{(i)}) = \Pi$. The bargaining set is a natural payoff solution concept to study the solutions J_0 and K_0 for the following reasons:

(i) the bargaining set for each c.s. consists of payoffs that are stable in the sense of objections and counter-objections. If for a particular c.s., a payoff is not in the bargaining set, some player would have a justified objection (an objection that has no counter-objection) which when carried out would result in breakup of the coalition structure. Hence we are not justified in using unstable payoffs corresponding to a c.s. to dominate another c.s. Also,

(ii) the bargaining set is nonempty for each coalition structure.

We shall now determine $K_0(M_1^{(i)})$ for all 3-person games with side payments.

Consider the 3-person game given by $N = \{1,2,3\}$,

$$(3.33)\qquad \begin{array}{l} v(1) = v(2) = v(3) = 0, \quad v(12) = a, \quad v(13) = b, \quad v(23) = c, \\ \text{and } v(123) = d, \text{ where } 0 \le a \le b \le c \text{ and } d \ge 0. \end{array}$$

Theorem 3.28. Let Γ be a 3-person game as in (3.33) with $d > (a+b+c)/2$.

(i) If $d < c$, then $K_0(M_1^{(i)}) = \{(1)(23)\}$

(ii) If $d = c$, then $K_0(M_1^{(i)}) = \{(1)(23), (123)\}$

(iii) If $d > c$, then $K_0(M_1^{(i)}) = \{(123)\}$.

Proof: (i) In this case we have $(a+b)/2 + c/2 < d < c/2 + c/2$, hence $a+b < c$. The bargaining set is given by

$$(3.34)\qquad M_1^{(i)}(P) = \begin{cases} (0,\ 0,\ 0) & \text{if } P = (1)(2)(3), \\ (0,\ a,\ 0) & \text{if } P = (12)(3), \\ (0,\ 0,\ b) & \text{if } P = (13)(2), \\ \mathrm{Conv}\{(0,\ c-b,\ b),\ (0,\ a,\ c-a)\} & \text{if } P = (1)(23), \\ (0,\ d/2 - (b-a)/2,\ d/2 + (b-a)/2) & \text{if } P = (123) \end{cases}$$

Clearly $(1)(23)$ dom$(M_1^{(i)})$ $(12)(3)$ and $(1)(23)$ dom$(M_1^{(i)})$ $(13)(2)$. Also since $(0,\ c/2 - (b-a)/2,\ c/2 + (b-a)/2) \in M_1^{(i)}((1)(23))$ and $c > d$, $(1)(23)$ dom$(M_1^{(i)})$ (123). The transition graph is shown in Figure 3.6. Hence Case (i) follows.

(ii) In this case, the bargaining set is as in (3.34) except for c.s. (123) which is

$$M_1^{(i)}((123)) = M_1^{(i)}((1)(23)).$$

Therefore (ii) follows. (See Figure 3.7.)

(iii) Case 1) $c > a+b$

Here the bargaining set is as in (3.34) except for c.s. (123) which is given by

$$M_1^{(i)}((123)) = \{(x_1,x_2,x_3)\colon x_1 + x_2 \geq a,\ x_1 + x_3 \geq b,\ x_2 + x_3 \geq c, \text{ and } x_1 + x_2 + x_3 = d\}.$$

For each $(0, x_2, c-x_2) \in M_1^{(i)}((1)(23))$ where $a \leq x_2 \leq c-b$, we have $((d-c)/3,\ x_2 + (d-c)/3,\ c-x_2 + (d-c)/3) \in M_1^{(i)}((123))$. Hence (123) $\mathrm{dom}(M_1^{(i)})$ (1)(23). The transition digraph is shown in Figure 3.8.

Case 2) $c \leq a+b$

In this case the bargaining set is given by

$$(3.35) \qquad M_1^{(i)}(P) = \begin{cases} (0,\ 0,\ 0) & \text{if } P = (1)(2)(3), \\ (p_1,\ p_2,\ 0) & \text{if } P = (12)(3), \\ (p_1,\ 0,\ p_3) & \text{if } P = (13)(2), \\ (0,\ p_2,\ p_3) & \text{if } P = (1)(23), \\ \mathrm{Co}((123)) & \text{if } P = (123). \end{cases}$$

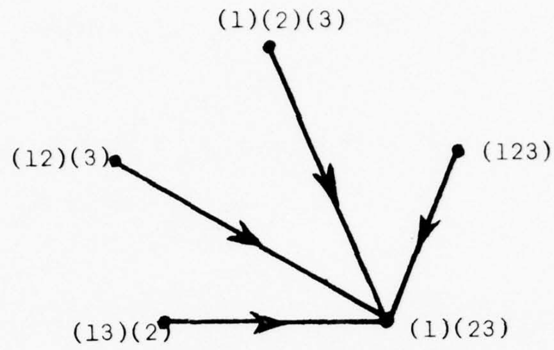


Figure 3.6. The transition digraph in Theorem 3.28, (i).

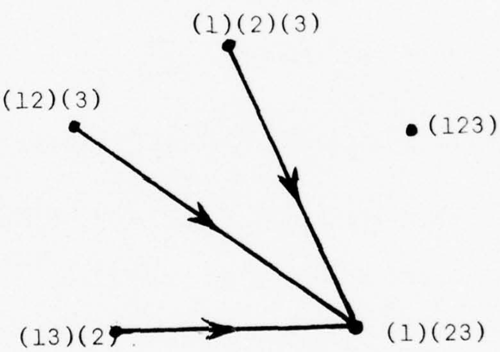


Figure 3.7. The transition digraph in Theorem 3.28, (ii).

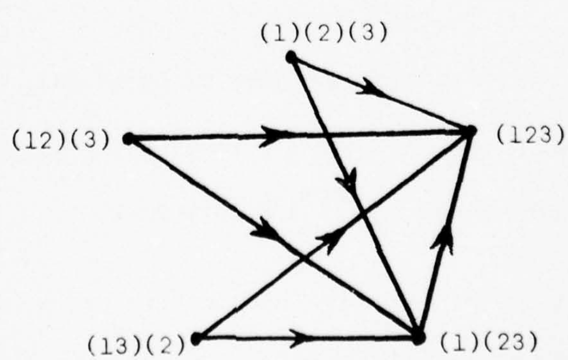


Figure 3.8. The transition digraph in Theorem 3.28, (iii) case 1).

where $p_1 = (a+b-c)/2$, $p_2 = (a+c-b)/2$, $p_3 = (b+c-a)/2$, and

$$Co((123)) = \{(x_1,x_2,x_3)\colon x_1 + x_2 \geq a,\ x_1 + x_3 \geq b,\ x_2 + x_3 \geq c, \text{ and } x_1 + x_2 + x_3 = d\}.$$

Let $p = (p_1 + p_2 + p_3)$, then clearly,

$$(p_1 + (d-p)/3,\ p_2 + (d-p)/3,\ p_3 + (d-p)/3) \in M_1^{(i)}((123))$$

Hence c.s. (123) dominates (w.r.t. $M_1^{(i)}$) every other c.s. This case completes the proof of the theorem. $\square$

Theorem 3.29. Let Γ be a 3-person game as in (3.33) with $d = (a+b+c)/2$.

(i) If $c \leq a+b$ then $K_0(M_1^{(i)}) = \{(12)(3), (13)(2), (1)(23), (123)\}$.

(ii) If $c > a+b$ then $K_0(M_1^{(i)}) = \{(1)(23)\}$.

Proof: (i) In this case, the bargaining set is as in (3.35) with $M_1^{(i)}((123)) = (p_1, p_2, p_3)$. The result clearly follows.

(ii) In this case, the bargaining set is as in (3.34). Since $d < c$, the result follows. $\square$

Theorem 3.30. Let Γ be a 3-person game as in (3.33), with $d < (a+b+c)/2$.

(i) If $c \leq a+b$ then $K_0(M_1^{(i)}) = \{(12)(3), (13)(2), (1)(23)\}$.

(ii) If $c > a+b$ then $K_0(M_1^{(i)}) = \{(1)(23)\}$.

Proof: (i) In this case, the bargaining set is as in (3.35) except for c.s. (123) for which it is given by

$$
(3.36)\quad M_1^{(i)}((123)) = \begin{cases} (p_1 + (d-p)/3,\ p_2 + (d-p)/3,\ p_3 + (d-p)/3) & \text{if } 2c-a-b \le d \\ (0,\ d/2 - (b-a)/2,\ d/2 + (b-a)/2) & \text{if } b-a \le d < 2c-a-b \\ (0,\ 0,\ d) & \text{if } d < b-a. \end{cases}
$$

In all cases, the transition graph is presented in Figure 3.9. Therefore (i) follows.

(ii) In this case the bargaining set is as in (3.34) except for c.s. (123) for which the bargaining set is as in (3.36). The transition graph is shown in Figure 3.10. Hence the result follows. □

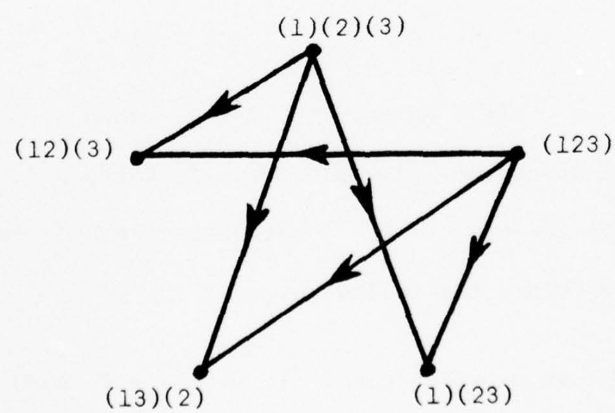


Figure 3.9. The transition graph in Theorem 3.30, (i).

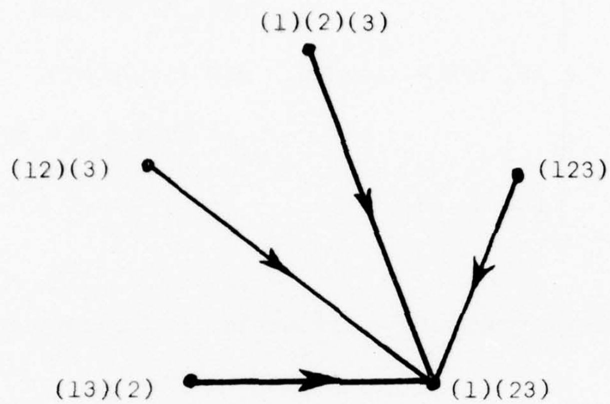


Figure 3.10. The transition graph in Theorem 3.30, (ii).

Since Theorems 3.28, 3.29 and 3.30 cover all cases, we have proved the following.

Theorem 3.31. Let Γ be a 3-person game as in (3.33). Then $K_0(M_1^{(i)}) \neq \emptyset$.

For every $P \in \Pi$, if $x \in I(P)$ belongs to $Co(P)$, then no player can have an objection against another player. Thus if $Co(P) \neq \emptyset$, $Co(P) \subset M_1^{(i)}(P)$. Hence the p.s.c. $M_1^{(i)}$ satisfies the hypothesis of Corollary 3.11. So we obtain the following.

Lemma 3.32. Let Γ be an n-person game. If $\Pi(Co) \neq \emptyset$ then $K_0(M_1^{(i)}) \neq \emptyset$. In fact $K_0(M_1^{(i)}) \supset \Pi_z$.

Proof: This is a consequence of Corollary 3.11 and Theorem 3.12. $\square$

No general existence theorem for $K_0(M_1^{(i)})$ is known at this time. Example 3.12 illustrates a pathology for $K_0(M_1^{(i)})$ which is due to

a "flaw" in the properties of the bargaining set.

Example 3.12. Let Γ be a 5-person game with

$$v(12) = 10, \quad v(35) = 85, \quad v(134) = 148, \quad v(2345) = 160, \quad \text{and}$$
$$v(R) = 0 \quad \text{for all other } R \subset N.$$

A simple computation reveals that the bargaining set is given by

$$M_1^{(i)}(P) = \begin{cases} (0, 10, 0, 0, 0) & \text{if } P = (12)(3)(4)(5),\ (12)(3)(45), \\ & \qquad (12)(345) \text{ or } (12)(34)(5), \\ (0, 0, 85, 0, 0) & \text{if } P = (1)(2)(35)(4),\ (14)(35)(2), \\ & \qquad (124)(35) \text{ or } (1)(24)(35), \\ (0, 0, 148, 0, 0) & \text{if } P = (134)(2)(5) \text{ or } (134)(25), \\ (0, 10 \le x_2 \le 12, 160\text{-}x_2, 0, 0)^{\dagger} & \text{if } P = (1)(2345), \\ (0, 10, 85, 0, 0) & \text{if } P = (12)(35)(4), \\ (0, 0, 0, 0, 0) & \text{for all other } P \in \Pi. \end{cases}$$

Note that in every c.s. that contains a coalition which has a positive value, at least one player in the coalition gets zero payoff in the bargaining set. As a result, due to Condition 3.5 in the definition of domination, no c.s. dominates another c.s. Hence $K_0(M_1^{(i)}) = \Pi$.

The above example exhibits a flaw in the properties of the bargaining set. E.g., in the c.s. (12)(35)(4) player 5 gets zero payoff in the

---

† Denotes the set $\{(0, x_2, 160\text{-}x_2, 0, 0)\colon 10 \le x_2 \le 12\}$.

bargaining set. This is because player 5 has no 'bargaining power' at all vis-á-vis player 3. Since there are no coalitions with a positive value that contains player 5 but not player 3, player 5 cannot even object! However the payoff in the bargaining is counter-intuitive because we could argue: Why should player 5 enter into a coalition with player 3 if his share of the resulting coalitional value is the same as what the player could have obtained had he been in a coalition by himself? In this respect, we could say that the bargaining set is derived entirely from the bargaining positions of the players in the process of coalition formation in contrast with the Shapley value which is derived entirely from the characteristic function of the game. These two p.s.c.'s reflect two extreme view points in looking at solutions of cooperative games in characteristic function form. A major research problem is to define a p.s.c. that exhibits both the strategic value and the bargaining power of the players.

One method of attacking this problem in the case of the bargaining set is to regard the bargaining set as an idealization (of the bargaining process) and relax the definition of an objection by ε, where ε is a small positive real number. More formally, let $x \in I(P)$ and i and j be two distinct players in a coalition $P_k \in P$. An ε-objection of i against j is a vector y^R, where R is a coalition containing player i but not j, whose coordinates y_ℓ satisfy $y_i > x_i + \varepsilon$, $y_\ell \geq x_\ell$ for all $\ell \in R$, and $\sum_{\ell \in R} y_\ell = v(R)$. A counter-objection to this ε-objection is defined as before. We say $x \in I(P)$ is ε-stable if for each ε-objection in x, there is a counter-objection. The ε-bargaining set, denoted by $M^{(i)}_{1,\varepsilon}$, corresponding to c.s. $P \in \Pi$ is the set of all ε-stable $x \in I(P)$, i.e.,

$$M^{(i)}_{1,\varepsilon}(P) = \{x \in I(P)\colon x \text{ is } \varepsilon\text{-stable}\}. \tag{3.37}$$

We could regard ε as a 'sacrifice' each player is willing to make (if necessary) for coalitional stability.

Note that the results in Theorems 3.28, 3.29, 3.30 and 3.31 as well as Lemma 3.32 remain unchanged if we replace $M^{(i)}_1$ by $M^{(i)}_{1,\varepsilon}$.

Example 3.13. Consider the game in Example 3.12. The ε-bargaining set is given by

$$M^{(i)}_{1,\varepsilon}(P) = \begin{cases} (0 \le x_1 \le \varepsilon,\ 10-x_1,\ 0,\ 0,\ 0) & \text{if } P = (12)(3)(4)(5), \\ & \quad (12)(3)(45),\ (12)(345) \text{ or } (12)(34)(5), \\ (0,\ 0,\ 85-x_5,\ 0,\ 0 \le x_5 \le \varepsilon) & \text{if } P = (1)(2)(35)(4), \\ & \quad (14)(2)(35),\ (124)(35) \text{ or } (1)(24)(35), \\ (0 \le x_1 \le \varepsilon,\ 10-x_1,\ 85-x_5,\ 0,\ 0 \le x_5 \le \varepsilon) & \text{if } P = (12)(35)(4), \\ (0 \le x_1 < \varepsilon,\ 0,\ 148-x_1-x_4,\ 0 \le x_4 \le \varepsilon,\ 0) & \text{if } P = (134)(25) \text{ or} \\ & \quad (134)(2)(5), \\ (0,\ 10-\varepsilon \le x_2 \le 12+\varepsilon,\ 160-x_2-x_4-x_5,\ 0 \le x_4 \le \varepsilon,\ 0 \le x_5 \le \varepsilon) & \\ & \text{if } P = (1)(2345), \\ (0,\ 0,\ 0,\ 0,\ 0) & \text{for all other } P \in \Pi. \end{cases}$$

It is clear that $K_0(M^{(i)}_{1,\varepsilon}) = \{(12)(35)(4),\ (134)(2)(5),\ (134)(25),\ (1)(2345)\}$ which is more intuitive than $K_0(M^{(i)}_1) = \Pi$.

Example 3.14. (The Chemical Company Game. See Anderson and Traynor [2].) Two chemical companies C_1 and C_2 supply two fabricating companies

F_1 and F_2. The permissible coalition structures are:

$$P_1 = (C_1)(C_2)(F_1)(F_2), \quad P_2 = (C_1F_1)(C_2)(F_2),$$
$$P_3 = (C_1F_2)(C_2)(F_1), \quad P_4 = (C_1)(C_2F_1)(F_2),$$
$$P_5 = (C_1)(C_2F_1)(F_1), \quad P_6 = (C_1F_1)(C_2F_2),$$
$$P_7 = (C_1F_2)(C_2F_1).$$

The respective payoffs (profits) to these coalitions in the particular coalition structures are:

P_1: 25, 15, 75, 100. P_2: 300, 25, 110.
P_3: 500, 30, 85. P_4: 28, 200, 105.
P_5: 30, 425, 90. P_6: 400, 600.
P_7: 700, 300.

This "partition function" induces the characteristic function:

$$v(C_1) = 25, \quad v(C_2) = 15, \quad v(F_1) = 75, \quad v(F_2) = 100, \quad v(C_1,F_1) = 300,$$
$$v(C_1,F_2) = 500, \quad v(C_2,F_1) = 200, \quad v(C_2,F_2) = 425.$$

The bargaining set $M_1^{(i)}$ is given by

$$M_1^{(i)}(P) = \begin{cases} (25,\ 15,\ 75,\ 100) & \text{if } P = P_1 \\ (115 \leq x_1 \leq 225,\ 15,\ 300-x_1,\ 100) & \text{if } P = P_2 \\ (90 \leq x_1 \leq 225,\ 15,\ 75,\ 500-x_1) & \text{if } P = P_3 \\ (25,\ 15 \leq x_2 \leq 125,\ 200-x_2,\ 100) & \text{if } P = P_4 \end{cases}$$

$$M_1^{(i)}(P) = \begin{cases} (25,\ 15 \le x_2 \le 125,\ 75,\ 425-x_2) & \text{if } P = P_5 \\ (x_1,\ x_2,\ 300-x_1,\ 425-x_2) & \text{if } P = P_6 \\ \quad \text{where } x_1, x_2 \text{ are as in Figure 3.11} & \\ (y_1,\ y_2,\ 200-y_2,\ 500-y_1) & \text{if } P = P_7 \\ \quad \text{where } y_1, y_2 \text{ are as in Figure 3.12.} & \end{cases}$$

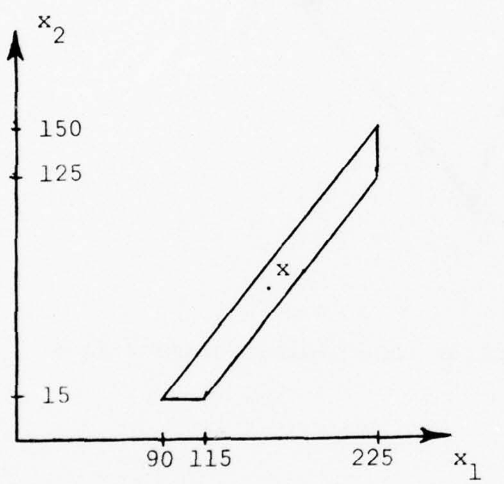


Figure 3.11. The bargaining set $M_1^{(i)}(P_6)$ for the chemical company game.

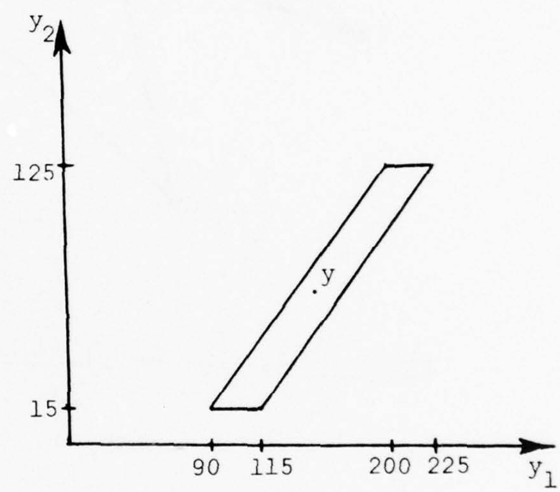


Figure 3.12. The bargaining set $M_1^{(i)}(P_7)$ for the chemical company game.

The transition digraph is shown in Figure 3.13. Hence $K_0(M_1^{(i)}) = \{(C_1F_1)(C_2F_2),\ (C_1F_2)(C_2F_1)\}$.

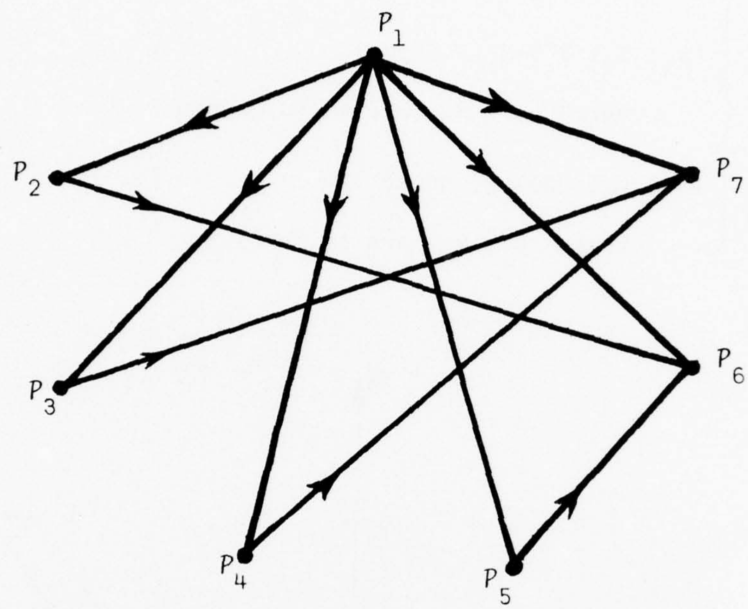


Figure 3.13. The transition digraph of the chemical company game.

3.8 Some Modifications of the Coalition Structure Model

In this section, we look at some modifications of the domination relation in the abstract game $(\Pi(S), \mathrm{dom}(S))$. We define two other domination relations one of which is stronger than $\mathrm{dom}(S)$ and the other weaker than $\mathrm{dom}(S)$.

Definition 3.9. Let $P_1, P_2 \in \Pi(S)$ and S be a p.s.c. Then P_1 weakly dominates P_2, denoted by P_1 w-dom(S) P_2, iff

(3.38) for each $y \in S(P_2)$, $\exists$ a nonempty $R \in P_1$ and $x \in S(P_1)$ such that $x_i > y_i$ for all $i \in R$.

Definition 3.10. Let $P_1, P_2 \in \Pi(S)$ and S be a p.s.c. Then P_1 strongly dominates P_2, denoted by P_1 s-dom(S) P_2, iff $\exists$ a nonempty $R \in P_1$ and $x \in S(P_1)$ such that for all $y \in S(P_2)$, $x_i > y_i$ for all $i \in R$.

The following relations are direct consequences of Definitions 3.6, 3.9 and 3.10.

(3.39) If P_1 s-dom(S) P_2, then P_1 dom(S) P_2.

(3.40) If P_1 dom(S) P_2, then P_1 w-dom(S) P_2.

Let $K_{0,w}(S)$ and $K_{0,s}(S)$ denote the cores of the abstract games $(\Pi(S),\text{w-dom}(S))$ and $(\Pi(S),\text{s-dom}(S))$ respectively. As a consequence of Relations (3.39) and (3.40), we have

$$K_{0,s}(S) \supset K_0(S) \supset K_{0,w}(S). \tag{3.41}$$

Also, if S is a p.s.c. such that for each $P \in \Pi$, $S(P)$ is either a single point set in E^n or an empty set, then

$$K_{0,s}(S) = K_0(S) = K_{0,w}(S).$$

CHAPTER IV

A COMPARISON WITH CAPLOW'S AND GAMSON'S THEORIES OF COALITION FORMATION

4.1 Introduction

In this chapter, we reformulate Caplow's and Gamson's theories of coalition formation in a more general and mathematical setting and compare the predictions of these approaches with our models. Caplow's theory is restricted to triads, i.e., a three person weighted majority game with a simple majority quota. Gamson's theory is applicable to all proper weighted majority games without dictators or veto players.

Before we make this comparison, we note that our theory is normative, whereas both Caplow's and Gamson's theories are descriptive. Like game theory, our theory is based on the assumption of "strict rationality". Luce and Raiffa write:

> "...it is crucial that social scientists recognize that game theory is not descriptive but rather (conditionally) normative. It states neither how people do behave nor how they should behave in an absolute sense, but how they should behave if they wish to achieve certain ends." [66]

However, as noted by Gamson [34, p. 380],

> "...a normative theory often provides a useful starting point for a descriptive theory."

4.2 Caplow's Theory of Coalitions in the Triad

Much of the recent research on coalition formation in sociology and psychology was generated by a paper by Caplow [22]. Caplow proposes that

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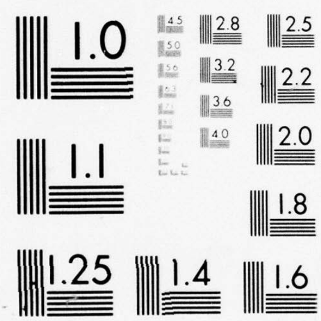
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Triad Type	Distribution of Resources	Equivalent Weighted Majority Representation	Predictions	
			Caplow	$K_0(\kappa)$
1	A = B = C	[2; 1,1,1]	(AB)(C), (AC)(B), (A)(BC)	(AB)(C), (AC)(B), (A)(BC)
2	A > B, B = C, A < (B+C)	[4; 3,2,2]	(A)(BC)	(A)(BC)
3	A < B, B = C	[4; 1,2,2]	(AB)(C), (AC)(B)	(AB)(C), (AC)(B)
4	A > (B+C), B = C	[3; 3,1,1]	(A)(B)(C)	(A)(B)(C), (A)(BC), (ABC)
5	A > B > C, A < (B+C)	[5; 4,3,2]	(AC)(B), (A)(BC)	(AC)(B), (A)(BC)
6	A > B > C, A > (B+C)	[4; 4,2,1]	(A)(B)(C)	(A)(B)(C), (A)(BC)
7	A > B > C, A = (B+C)	[4; 3,2,1]	(AB)(C), (AC)(B)	(AB)(C), (AC)(B), (ABC)
8	A = (B+C), B = C	[3; 2,1,1]	(AB)(C), (AC)(B)	(AB)(C), (AC)(B), (ABC)

Table 4.1

A comparison of Caplow's predictions with $K_0(\kappa)$.

the formation of coalitions

> "depends upon the initial distribution of power, and, other things being equal, may be predicted under certain assumptions when the initial distribution of power is known." [22]

Caplow's four assumptions are:

A.1. Members of a triad may differ in strength. A stronger member can control a weaker member and will seek to do so.

A.2. Each member of the triad seeks control over the others. Control over two others is preferred to control over one other. Control over one other is preferred to control over none.

A.3. Strength is additive. The strength of a coalition is equal to the sum of the strengths of its two members.

A.4. The formation of coalitions takes place in an existing triadic situation, so that there is a pre-coalition condition in every triad. Any attempt by a stronger member to coerce a weaker member into joining a non-advantageous coalition will provoke the formation of an advantageous coalition to oppose the coercion.

Caplow enumerates six different triadic power structures and, based on his assumptions, makes predictions as to which coalitions will form in each type of triad. In a subsequent paper, Caplow [23] lists two more types of triads that were overlooked in the original presentation along with his predictions. The predictions are listed in Table 4.1. Before we compare our theories with Caplow's theory, we

will restate Caplow's theory in a mathematical setting[†].

Let Γ be an n-person weighted majority game

$$(4.1) \qquad [q; a_1,\ldots,a_n] \quad \text{where} \quad q > (a_1 + \ldots + a_n)/2,$$

and let $\mathcal{W}$ denote the set of all winning coalitions in Γ. Let i and j be two distinct players. We say that <u>player</u> i <u>controls</u> <u>player</u> j in <u>coalition</u> <u>structure</u> $\mathcal{P}$ iff either

$$(4.2) \qquad a_i > a_j, \text{ and } i,j \in P_k \in \mathcal{W}, \quad P_k \in \mathcal{P}, \text{ or}$$

$$(4.3) \qquad i \in P_k \in \mathcal{W}, \quad j \notin P_k, \quad P_k \in \mathcal{P}.$$

Let $\beta(\mathcal{P})(i)$ denote the number of players player i controls in c.s. $\mathcal{P}$. The <u>Caplow</u> <u>Power</u> <u>Index</u>, denoted by κ, is defined as follows:

$$(4.4) \qquad \kappa(\mathcal{P})(i) = \begin{cases} \beta(\mathcal{P})(i)/\sum\limits_{j\in N} \beta(\mathcal{P})(j) & \text{if } \sum\limits_{j\in N} \beta(\mathcal{P})(j) \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

for all $i \in N$ and all $\mathcal{P} \in \Pi$.

Intuitively, $\kappa(\mathcal{P})(i)$ denotes the relative power of player i when the players are aligned as in c.s. $\mathcal{P}$[††].

[†]The author assumes full responsibility for the ensuing formulation, which, though never formally stated, is implicit in Caplow's paper [22].

[††]Note that, although Caplow stated his theory only for the restricted case of triads, our formulation of Caplow's theory holds for the more general case of n-person proper weighted majority games.

We are now in a position to compare Caplow's predictions with the predictions of our theories. Since a unique outcome is associated with each coalition structure, by Theorem 3.6, the s.c. model and the c.s. model indicate the same results with respect to the Caplow power index. Examples 4.1-4.8 deal with the eight different types of triads analyzed by Caplow. At the end of each example, we quote Caplow's analysis of the triad, partly to justify our definition of the Caplow power index.

<u>Example 4.1</u>. Consider the Type 1 triad $[2;\ \underset{A}{1},\underset{B}{1},\underset{C}{1}]$. Then the Caplow power index, κ, is given by

$$\kappa(P) = \begin{cases} (0,\ 0,\ 0) & \text{if } P = (A)(B)(C) \\ (1/2,\ 1/2,\ 0) & \text{if } P = (AB)(C) \\ (1/2,\ 0,\ 1/2) & \text{if } P = (AC)(B) \\ (0,\ 1/2,\ 1/2) & \text{if } P = (A)(BC) \\ (0,\ 0,\ 0) & \text{if } P = (ABC) \end{cases}$$

The transition digraph is as in Figure 4.1. $K_0(\kappa) = \{(AB)(C), (AC)(B), (A)(BC)\}$. Caplow argues:

> "...each member strives to enter a coalition within which he is equal to his ally and stronger (by virtue of the coalition) than the isolate." [22]

<u>Example 4.2</u>. Consider the Type 2 triad $[5;\ \underset{A}{3},\underset{B}{2},\underset{C}{2}]$. Then the Caplow power index, κ, is given by

$$\kappa(P) = \begin{cases} (0,\ 0,\ 0) & \text{if } P = (A)(B)(C) \\ (2/3,\ 1/3,\ 0) & \text{if } P = (AB)(C) \\ (2/3,\ 0,\ 1/3) & \text{if } P = (AC)(B) \\ (0,\ 1/2,\ 1/2) & \text{if } P = (A)(BC) \\ (1,\ 0,\ 0) & \text{if } P = (ABC) \end{cases}$$

The transition digraph is shown in Figure 4.2. $K_0(\kappa) = \{(A)(BC)\}$.

Caplow argues:

> "...Consider the position of B. If he forms a coalition with A, he will (by virtue of the coalition) be stronger than C, but within the coalition he will be weaker than A. If, on the other hand, he forms a coalition with C, he will be equal to C within the coalition and stronger than A by virtue of the coalition. The position of C is identical with that of B." [22]

Example 4.3. Consider the Type 3 triad $[3;\ \underset{A}{1},\underset{B}{2},\underset{C}{2}]$. Then the Caplow power index, κ, is given by

$$\kappa(P) = \begin{cases} (0,\ 0,\ 0) & \text{if } P = (A)(B)(C) \\ (1/3,\ 2/3,\ 0) & \text{if } P = (AB)(C) \\ (1/3,\ 0,\ 2/3) & \text{if } P = (AC)(B) \\ (0,\ 1/2,\ 1/2) & \text{if } P = (A)(BC) \\ (0,\ 1/2,\ 1/2) & \text{if } P = (ABC) \end{cases}$$

The transition digraph is shown in Figure 4.3. $K_0(\kappa) = \{(AB)(C),\ (AC)(B)\}$.

Caplow argues:

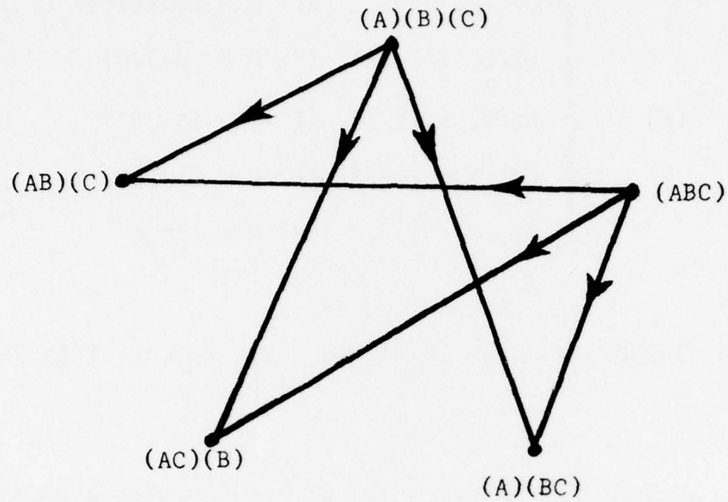

Figure 4.1. The transition digraph of Type 1 triad.

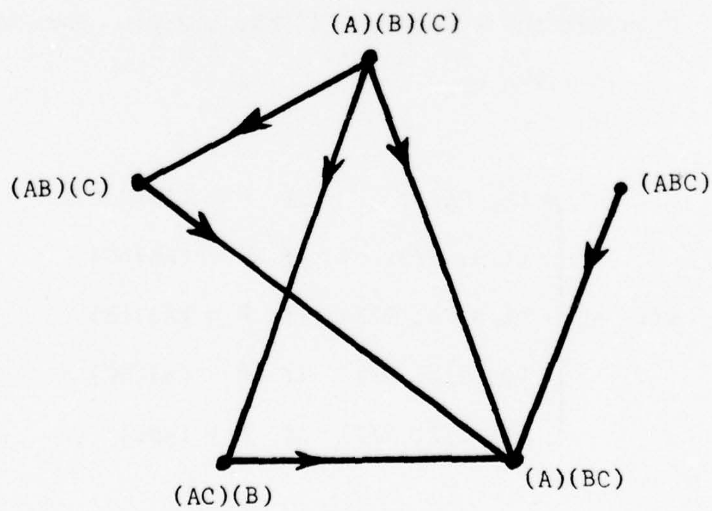

Figure 4.2. The transition digraph of Type 2 triad.

> "...A may strengthen his position by forming a coalition with either B or C, and will be welcomed as an ally by either B or C. On the other hand, if B joins C, he does not improve his pre-coalition position of equality with C and superiority to A. His only motive to enter a coalition with C is to block AC. However, C's position is identical with B and he, too, will prefer A to B as an ally." [22]

Example 4.4. Consider the Type 4 triad $[3; \underset{A}{3}, \underset{B}{1}, \underset{C}{1}]$. Then the Caplow power index, κ, is given by

$$\kappa(P) = \begin{cases} (1, 0, 0) & \text{if } P = (A)(B)(C) \\ (2/3, 1/3, 0) & \text{if } P = (AB)(C) \\ (2/3, 0, 1/3) & \text{if } P = (AC)(B) \\ (1, 0, 0) & \text{if } P = (A)(BC) \\ (1, 0, 0) & \text{if } P = (ABC) \end{cases}$$

The transition digraph is shown in Figure 4.4. $K_0(\kappa) = \{(A)(B)(C), (A)(BC), (ABC)\}$. Caplow argues:

> "...B and C have no motive to enter a coalition with each other. Once formed, the coalition would still be weaker than A and they would still be equal within it. A on the other hand, has no motive to form a coalition with B or C, since he is stronger than each of them and is not threatened by their coalition. No coalition will be formed, unless B or C can find some extraneous means of inducing A to join them." [22]

Example 4.5. Consider the Type 5 triad $[5; \underset{A}{4}, \underset{B}{3}, \underset{C}{2}]$. Then the Caplow power index, κ, is given by

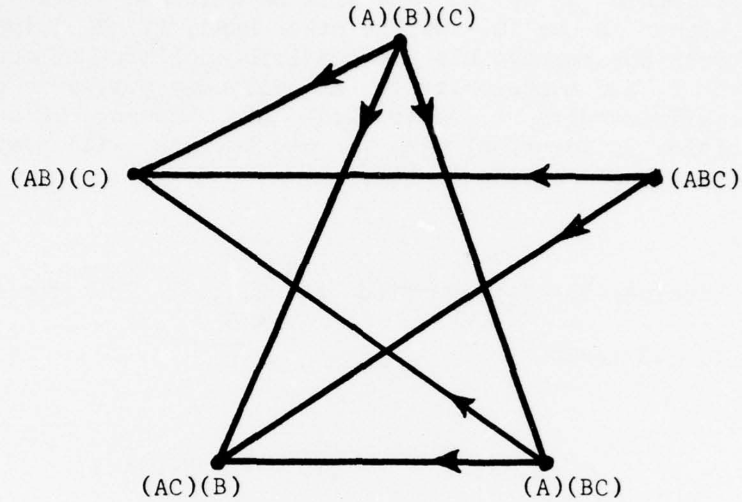

Figure 4.3. The transition digraph of Type 3 triad.

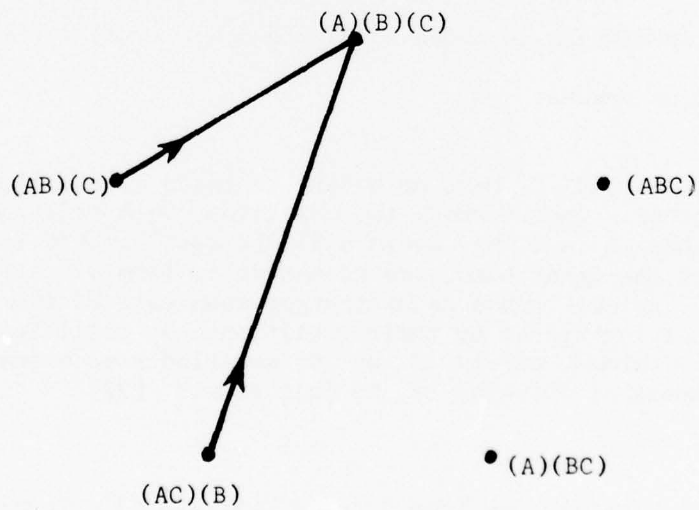

Figure 4.4. The transition digraph of Type 4 triad.

$$\kappa(P) = \begin{cases} (0,\ 0,\ 0) & \text{if } P = (A)(B)(C) \\ (2/3,\ 1/3,\ 0) & \text{if } P = (AB)(C) \\ (2/3,\ 0,\ 1/3) & \text{if } P = (AC)(B) \\ (0,\ 2/3,\ 1/3) & \text{if } P = (A)(BC) \\ (2/3,\ 1/3,\ 0) & \text{if } P = (ABC) \end{cases}$$

The transition digraph is shown in Figure 4.5. $K_0(\kappa) = \{(AC)(B), (A)(BC)\}$.

Caplow argues:

> "...A seeks to join both B and C and C seeks to join both A and B but B has no incentive to enter a coalition with A and has a very strong incentive to enter a coalition with C. Whether the differential strength of A and B will make them differentially attractive to C lies outside the scope of our present assumptions." [22]

<u>Example 4.6</u>. Consider the Type 6 triad $[4;\ \underset{A}{4},\underset{B}{2},\underset{C}{1}]$. Then the Caplow power index, κ, is given by

$$\kappa(P) = \begin{cases} (1,\ 0,\ 0) & \text{if } P = (A)(B)(C) \\ (2/3,\ 1/3,\ 0) & \text{if } P = (AB)(C) \\ (2/3,\ 0,\ 1/3) & \text{if } P = (AC)(B) \\ (1,\ 0,\ 0) & \text{if } P = (A)(BC) \\ (2/3,\ 1/3,\ 0) & \text{if } P = (ABC) \end{cases}$$

The transition digraph is as in Figure 4.6. $K_0(\kappa) = \{(A)(B)(C), (A)(BC)\}$.

Caplow argues:

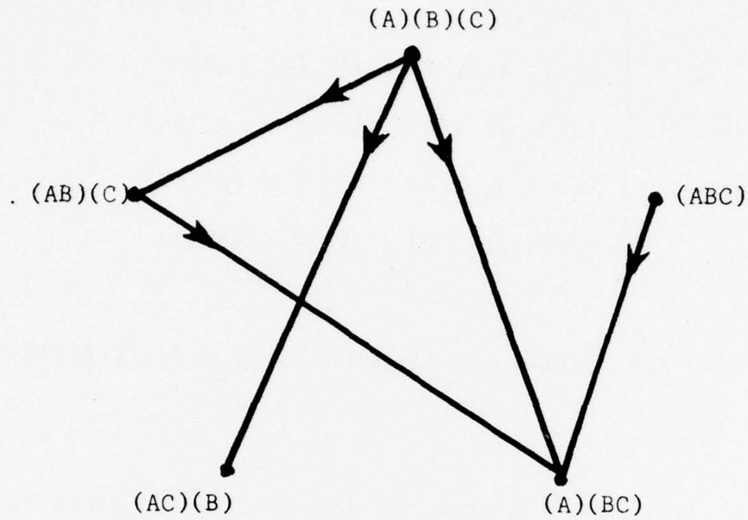

Figure 4.5. The transition digraph of Type 5 triad.

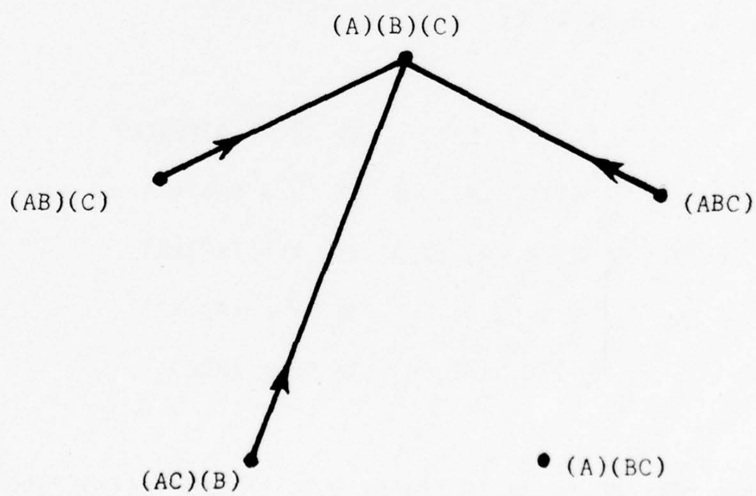

Figure 4.6. The transition digraph of Type 6 triad.

> "...A is stronger than B and C combined and has no motive to form a coalition. As in Type 4, true coalition is impossible. However, while in Type 4 both of the weaker members seek to join the stronger member, only C can improve his position by finding some extraneous means of inducing A to join him." [22]

By claiming that only C can improve his position by joining A, Caplow seem to imply that B controls C in the c.s. (A)(B)(C). Such an assumption seems unreasonable to us and we resolve this small discrepancy by suggesting that Caplow has erred in making such a claim. Note that a similar discrepancy arises in Caplow's analysis of the Type 3 triad where he claims that B is superior to A in c.s. (A)(B)(C).

Example 4.7. Consider the Type 7 triad $[4;\ \underset{A}{3},\underset{B}{2},\underset{C}{1}]$. Then the Caplow power index, κ, is given by

$$\kappa(P) = \begin{cases} (0,\ 0,\ 0) & \text{if } P = (A)(B)(C) \\ (2/3,\ 1/3,\ 0) & \text{if } P = (AB)(C) \\ (2/3,\ 0,\ 1/3) & \text{if } P = (AC)(B) \\ (0,\ 0,\ 0) & \text{if } P = (A)(BC) \\ (2/3,\ 1/3,\ 0) & \text{if } P = (ABC) \end{cases}$$

The transition digraph is shown in Figure 4.7. Hence, $K_0(\kappa) = \{(AB)(C), (AC)(B), (ABC)\}$.

Example 4.8. Consider the Type 8 traid $[3;\ \underset{A}{2},\underset{B}{1},\underset{C}{1}]$. Then the Caplow power index, κ, is given by

$$\kappa(P) = \begin{cases} (0,\ 0,\ 0) & \text{if } P = (A)(B)(C) \\ (2/3,\ 1/3,\ 0) & \text{if } P = (AB)(C) \\ (2/3,\ 0,\ 1/3) & \text{if } P = (AC)(B) \\ (0,\ 0,\ 0) & \text{if } P = (A)(BC) \\ (1,\ 0,\ 0) & \text{if } P = (ABC) \end{cases}$$

The transition digraph is as in Figure 4.7. Hence, $K_0(\kappa) = \{(AB)(C), (AC)(B), (ABC)\}$. For the Type 7 and 8 triads, Caplow argues:

> "...the combined strength of B and C is exactly equal to A, so that no effective coalition of B and C is strategically possible. In other words, although a coalition of B and C can block the dominance of A, it is not sufficient to control the situation, and, therefore, the probable coalitions under the standard assumptions are AB or AC." [23]

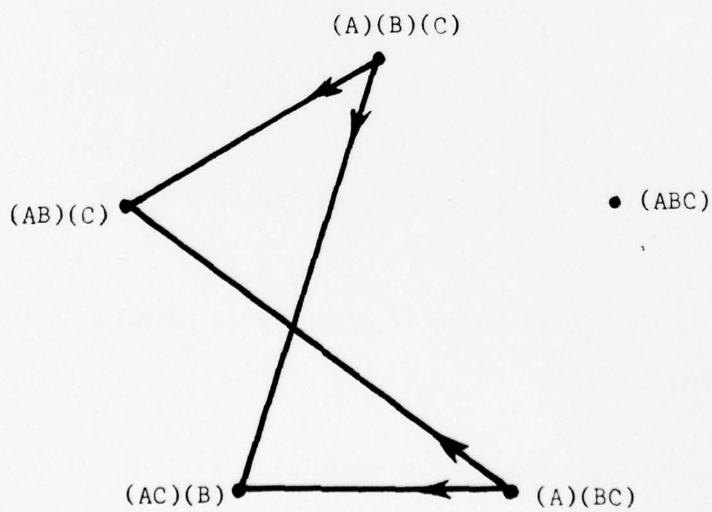

Figure 4.7. The transition digraph of Types 7 and 8 triads.

This completes our analysis of the eight different triads. The results are summarized in Table 4.1. A comparison reveals almost total agreement. All the c.s.'s predicted by Caplow are predicted by our theory. The only disagreements are in Types 4, 6, 7, 8, where our theory predicts more c.s.'s than that predicted by Caplow. However, this can easily be explained. Caplow implicitly assumes that in every triad, bargaining for coalitions start from the c.s. (A)(B)(C). A quick look at Figures 4.1-4.7 will reveal that with this additional assumption, our theory gives exactly the same predictions as Caplow's.

Vinacke and Arkoff [109] conducted experiments to test Caplow's theory. Their results, shown in Table 4.2, tend to support Caplow's theory in general with a few disagreements especially in the case of Type 3 and Type 5 triads. In the Type 3 triad, Caplow predicts coalition structures (AB)(C) and (AC)(B) without any reference to their relative frequency of occurrence. However Vinacke and Arkoff note that in the Type 3 triad, c.s. (AC)(B) occurs more frequently than c.s. (AB)(C). In the Type 5 triad, Caplow predicts coalition structures (AC)(B) and (A)(BC) with the reservation that

> "...whether the differential strength of A and B will make them differentially attractive to C lies outside the scope of our present assumptions." [22]

The results of the Vinacke-Arkoff experiments indicate that in the Type 5 triad, c.s. (A)(BC) occurs more often than c.s. (AC)(B).

Chertkoff [25] makes an additional assumption which leads to the conclusion that in the Type 5 triad, c.s. (A)(BC) occurs twice as frequently as (AC)(B) and that c.s. (AB)(C) does not occur at all.

Type	1	2	3	4	5	6
Equivalent Weighted Majority Representation / Coalition Structures	[2; 1,1,1]	[4; 3,2,2]	[3; 1,2,2]	[3; 3,1,1]	[5; 4,3,2]	[4; 4,2,1]
(A)(B)(C)	8	1	11	62	2	60
(AB)(C)	33	13	24	11	9	9
(AC)(B)	17	12	40	10	20	13
(A)(BC)	30	64	15	7	59	8
(ABC)	2	0	0	0	0	0
Total	90	90	90	90	90	90

Table 4.2

Coalition structures formed in the six types of triads in the Vinacke-Arkoff experiments.

Also, the same assumption when applied to the case of Type 3 triad leads to the conclusion that c.s.'s (AB)(C) and (AC)(B) are equally likely and c.s. (A)(BC) does not occur at all.

Let us assume that all transitions from each coalition structure are equally likely. Then given an initial probability distribution on the set of all viable coalition structures, we can compute the probability of formation of each coalition structure in $K_1(S)$. E.g., in the Type 5 triad, given that players start (with probability 1) from c.s. (A)(B)(C), we observe that (Figure 4.8) c.s. (AB)(C) forms with probability 1/3, c.s. (AC)(B) forms with probability 1/3 and c.s. (A)(BC) forms with probability 1/3. However, once c.s. (AB)(C) is formed, c.s. (A)(BC) occurs with probability 1. The net result is that c.s. (A)(BC) occurs with probability 2/3 and c.s. (AC)(B) occurs with probability 1/3. Coalition structure (AB)(C) also forms with probability 1/3 but only as an intermediate c.s., i.e., only temporarily.

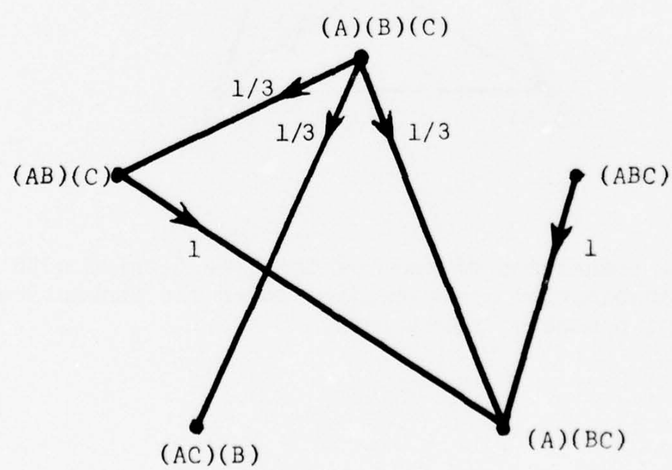

Figure 4.8. The transition digraph of the Type 5 triad with the probability of transitions under the assumption of equiprobable transitions.

A similar analysis of the Type 3 triad (Figure 4.9) indicates that, starting from c.s. (A)(B)(C) (with probability 1), c.s. (AB)(C) occurs with probability 1/2 and c.s. (AC)(B) occurs with probability 1/2. Coalition structure (A)(BC) occurs only as an intermediate coalition structure with probability 1/3. A summary of the predictions of our theories under the assumption of equi-probable transitions is shown in Table 4.3. Note that these predictions agree quite well with the Vinacke-Arkoff experimental results.

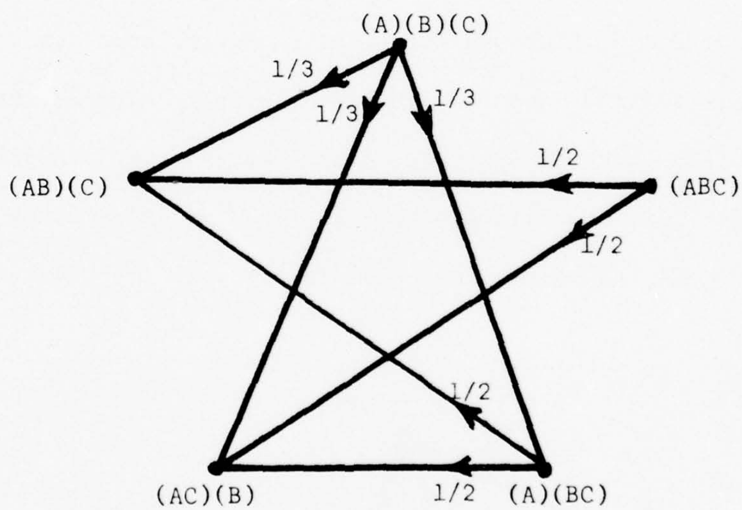

Figure 4.9. The transition digraph of the Type 3 triad with the probabilities of transition under the assumption of equi-probable transitions.

Triad Type	Equivalent Weighted Majority Representation	Starting Coalition Structure (assumed)	Probability	Intermediate Coalition Structures	Probability	Final Coalition Structures $K_1(\kappa)$	Probability
1	[2; 1,1,1]	(A)(B)(C)	1			(AB)(C) (AC)(B) (A)(BC)	1/3 1/3 1/3
2	[4; 3,2,2]	(A)(B)(C)	1	(AB)(C) (AC)(B)	1/3 1/3	(A)(BC)	1
3	[3, 1,2,2]	(A)(B)(C)	1	(A)(BC)	1/3	(AB)(C) (AC)(B)	1/2 1/2
4	[3; 3,1,1]	(A)(B)(C)	1			(A)(B)(C)	1
5	[5; 4,3,2]	(A)(B)(C)	1	(AB)(C)	1/3	(AC)(B) (A)(BC)	1/3 2/3
6	[4; 4,2,1]	(A)(B)(C)	1			(A)(B)(C)	1
7	[4; 3,2,1]	(A)(B)(C)	1			(AB)(C) (AC)(B)	1/2 1/2
8	[3; 2,1,1]	(A)(B)(C)	1			(AB)(C) (AC)(B)	1/2 1/2

Table 4.3

A summary of the predictions of the c.s. model under the assumption of equi-probable transitions.

4.3 Gamson's Theory of Coalition Formation

Following Caplow, Gamson formulated a slightly more general theory of coalition formation in proper weighted majority games without dictators or veto players. Before we present Gamson's theory, we need a definition. Let Γ be a weighted majority game. A cheapest winning coalition is a winning coalition whose total weight is a minimum among all winning coalitions. Gamson's main hypothesis is as follows:

> "Any participant will expect others to demand from a coalition a share of the payoff proportional to the amount of resources which they contribute to a coalition."

Here, a participant refers to a player, and his resources refers to his weight in the weighted majority game. Based on his main hypothesis, Gamson makes the following predictions about coalition formation.

(i) A player will favor a cheapest winning coalition.

(ii) A coalition of two distinct players $\{i,j\}$ will form if and only if there are reciprocal strategy choices between the two players. I.e. both player i and player j prefer coalition $\{i,j\}$.

(iii) The process of coalition formation is a step by step process where two players merge together into a coalition at a time.

(iv) Once a two-person coalition forms, the situation becomes a new one--the two players in the coalition are replaced by one player whose weight equals the sum of the weights of the two players in the coalition.

Implicit in Gamson's main hypothesis is a definition of a payoff value concept. Let $\Gamma = [q; a_1, \ldots, a_n]$ be a proper weighted majority game without a dictator or a veto player. Then the <u>Gamson power index</u>, denoted by γ, is given by

$$\gamma(P)(i) = \begin{cases} \dfrac{a_i}{\sum\limits_{i \in P_k} a_i} \cdot v(P_k) & \text{if } \sum\limits_{i \in P_k} a_i \neq 0 \\ 0 & \text{if } \sum\limits_{i \in P_k} a_i = 0 \end{cases} \tag{4.5}$$

where $P_k \in P$ is such that $i \in P_k$, for all $P \in \Pi$ and all $i \in N$. Let

$$g = \min_{R \in W} \sum_{i \in R} a_i \tag{4.6}$$

and

$$\Pi_g = \{P \in \Pi: \ P \text{ contains a cheapest winning coalition}\}. \tag{4.7}$$

Then Theorem 4.1 tells us what our model predicts using Gamson power index as a p.s.c.

<u>Theorem 4.1</u>. Let Γ be a proper weighted majority game. Then $K_0(\gamma) = \Pi_g$.

<u>Proof</u>: Let $P_1 \in \Pi_g$. Suppose $P_2 \in \Pi$ such that $P_2 \operatorname{dom}_R(\gamma) P_1$ for some $R \in P_2$ with $R \in W$. Then $\gamma(P_2)(i) > \gamma(P_1)(i)$ for all $i \in R$. Let $T \in P_1$ such that $T \in W$ and $\sum\limits_{i \in T} a_i = g$. Since Γ is proper, $R \cap T \neq \emptyset$. Let $j \in R \cap T$. Then $\gamma(P_1)(j) = a_j/g$. Since $j \in R$, $\gamma(P_2)(j) = a_j/(\sum\limits_{i \in R} a_i) > a_j/g$; i.e., $\sum\limits_{i \in R} a_i < g$ and a contradiction (from the definition of g) results. Hence $K_0(\gamma) \supset \Pi_g$.

Let $P_1 \in \Pi_g$ and $P_2 \in \Pi$ such that $P_2 \notin \Pi_g$. Then $P_1 \ \mathrm{dom}_T(\gamma) \ P_2$ where $T \in P_1$ such that $T \in \mathcal{W}$ and $\sum_{i \in T} a_i = g$, because $\gamma(P_1)(i) = a_i/g$ for all $i \in T$ and $\gamma(P_2)(i) < a_i/g$ for all $i \in T$. Hence $K_0(\gamma) \subset \Pi_g$. $\square$

It can be easily shown that Gamson's predictions (i)-(iv) about coalition formation lead to c.s.'s in Π_g. However Gamson assumes that players begin forming coalitions starting from one player coalitions. So if we choose only those c.s.'s in Π_g that are accessible from the c.s. consisting of only one player coalitions, our model reaches the same conclusions as Gamson's predictions.

CHAPTER V

A RESTRICTED BARGAINING SET

5.1 Introduction

In R. J. Aumann and M. Maschler [9], a theory was developed to attack the following general question: If the players in a cooperative n-person game have decided upon a specific coalition structure, how then will they distribute among themselves the values of the various coalitions in such a way that some stability requirements will be satisfied (cf. Davis and Maschler [29]). In this chapter, we do not assume that players have any a priori preference for any particular coalition structure. Assuming only rational behaviour, we study the outcomes of n-person cooperative games with side payments in terms of coalition structures and disbursement of payoffs that satisfy certain stability requirements. These stability requirements are modelled in the same manner as in the Aumann-Maschler (A-M) bargaining sets, centering upon the idea that a "stable" payoff configuration should offer some security in the sense that each "objection" could be met by a "counter objection."

In Section 5.2, we discuss some aspects of the A-M bargaining set $M_1^{(i)}$ which reflect the fact that a given coalition structure is assumed to be fixed and the bargaining is done under this assumption. In Section 5.3, a modification of the A-M bargaining set $M_1^{(i)}$ called the coalitional bargaining set, M_c, is introduced. Another bargaining set called the restricted bargaining set, M_r, is also defined. The restricted bargaining set is a subset of the coalitional bargaining set and results when the stability requirements in M_c are slightly

strengthened. Section 5.4 consists of a few examples which illustrate the basic differences between the bargaining sets M_r and $M_1^{(i)}$. Finally, in Section 5.5, the restricted bargaining set for all 3-person games with side payments is determined. Also a few general results are presented.

5.2 Some Comments on the Aumann-Maschler Bargaining Set $M_1^{(i)}$

The definitions of an objection and a counterobjection in the A-M bargaining set $M_1^{(i)}$ are made with the objective of identifying stable payoff configurations given that a particular coalition structure is assumed to be fixed. The reasons for this inference are as follows.

(i) In the definition of an objection, (see Chapter 3, Section 3.7), a player is allowed to object only against players in his own coalition. Hence the payoff configuration $((v(1),\ldots,v(n));\ (1)(2)\ldots(n))$ is trivially stable because each player is in a coalition by himself and has nobody to object against! This feature of an objection makes sense only if the coalition structure is assumed to be fixed whence a player can only object to the distribution of the payoff of his coalition.

(ii) Let (x,P) be a p.c., y^R be an objection by player i against player j in (x,P), and z^D be a counter-objection by j against i. We may have $R \cap D = \emptyset$ in which case, the only thing that prevents player i from carrying out his objection is the assumption that player i wishes to stick to coalition structure P (which would be destroyed if player i carries out his objection). Example 5.1 illustrates this fact.

Before we present the example, we introduce a definition. Let (N,v) be a game and $P = (P_1,\ldots,P_m)$ be a partition of N. The game (N,v) is said to be <u>decomposable with partition</u> P if for all $R \in 2^N$,

$$(5.1) \qquad v(R) = \sum_{j=1}^{m} v(S \cap P_j).$$

<u>Example 5.1</u>. Let $N = \{1,2,3,4,5,6\}$ and v be given by $v(i) = 0$ for all $i \in N$, $v(12) = v(13) = v(23) = 2$, $v(45) = v(46) = v(56) = 2$, $v(123) = 3$, $v(456) = 3$ and $v(R) = v(R\cap(123)) + v(R\cap(456))$ for all other $R \subset N$. Note that the game is decomposable with partition $(123)(456)$. Consider the p.c. $((1,1,1,2/3,2/3,2/3),(123456)) \in M_1^{(i)}$. An objection to this p.c. by player 4 (or 5 or 6) against player 1 (or 2 or 3) is $((1,1,1),(456))$. A counter objection to this objection by player 1 against player 4 is $((1,1),(12))$. Thus players 1, 2 and 3 are able to exploit the assumption that players 4, 5 and 6 wish to stick to the grand coalition, to their own (unfair) advantage. Without this assumption, there is nothing that player 1 (or 2 or 3) can do to stop player 4 (or 5 or 6) from carrying out the objection $((1,1,1),(456))$.

For the reasons outlined above, the A-M bargaining set $M_1^{(i)}$ assumes more than just rational behaviour on the part of the players and has to be interpreted as follows: Given that the players have decided upon a specific coalition structure P, $M_1^{(i)}(P)$ represents likely (probable) disbursements of payoffs which are stable in the sense that any objection by a player against another player in his coalition could be met with a counter objection. We would like to study the outcomes of games in terms of disbursement of payoffs and formation of coalitions under a

scheme of objections and counter objections assuming only rational behaviour of the players. I.e. we do not assume that players have any a priori preference for any coalition structure. This will be the subject of study in the subsequent sections of this chapter.

5.3 The Restricted Bargaining Set

Let $\Gamma = (N,v)$ be an n-person game with side payments.

The superadditive cover of a game (N,v) is the game $(N,\hat{v})$ defined by

$$(5.2) \qquad \hat{v}(R) = \max\{\sum_{i=1}^{p} v(R_i) : (R_1,\ldots,R_p) \text{ is a partition of } R\}.$$

Note that the superadditive cover of a game is itself superadditive. Also if v is superadditive and $\hat{v}$ is its superadditive cover, then $\hat{v} = v$. Even though some n-person game may not be superadditive, a coalition can always realize its value in the superadditive cover by suitably coordinating their strategies, i.e. by forming the partition that achieves the maximum value in Expression (5.2). For this reason, we will only deal with superadditive games for the rest of this chapter. However, nonsuperadditive games can be analyzed as follows. We study the restricted bargaining set for the superadditive cover. The results regarding coalition formation (as determined for the superadditive cover) are then "translated" back to the original nonsuperadditive game using Expression (5.2). We illustrate by means of an example.

Example 5.2. Let $N = \{1,2,3\}$ and v be given by $v(i) = 0$ for all $i \in N$, $v(12) = v(13) = 100$, $v(23) = 50$ and $v(123) = 0$.

The game is not superadditive. Its superadditive cover is given by

$$\hat{v}(123) = 100, \quad \hat{v}(R) = v(R) \quad \text{for all other} \quad R \subset N.$$

Suppose our theory, when applied to the game $(N,\hat{v})$, indicates that c.s. (123) shall form. Since

$$\hat{v}(123) = v(12) + v(3) = v(13) + v(2),$$

this corresponds to the statement that c.s.'s $(12)(3)$ or $(13)(2)$ will form in the game (N,v).

A <u>payoff configuration</u> (p.c.) is a pair (x,P) such that $x \in I(P)$, $P \in \Pi$ where $I(P)$ denotes the set of all individually rational payoffs as defined in Section 3.4, Chapter 3.

<u>Definition 5.1</u>. Let (x,P) be a p.c. for a game Γ where $P = (P_1,\ldots,P_m)$. Let R and T be coalitions for which

$$(5.3) \qquad \emptyset \neq R \subset N,$$

and

$$(5.4) \qquad T = \Big(\bigcup_{P_j \cap R \neq \emptyset} P_j \Big) - R.$$

A <u>coalitional objection</u> of R against T in (x,P) is a vector y^R for which

$$(5.5) \qquad y_i^R > x_i \quad \forall\, i \in R,$$

and

(5.6) $$\sum_{i \in R} y_i^R = v(R).$$

Thus, once a coalition decides to object, it has no choice of players against whom it is objecting, i.e. it cannot single out a particular coalition against whom the objection is directed. The objection is directed towards those players whose coalitions are disrupted by the objecting coalition R. Also we do not distinguish any particular player in R as making the objection.

Definition 5.2. Let (x,P) be a p.c. in a game Γ and let y^R be a coalitional objection of R against T in (x,P) where R and T satisfy (3.1). A coalitional counter objection of T against R is a vector z^D for which

(5.7) $$D \cap T \neq \emptyset,$$

(5.8) $$D \cap R \neq \emptyset, \quad D \neq R,$$

(5.9) $$z_i^D > y_i \quad \forall\, i \in D \cap R,$$

(5.10) $$z_i^D \geq x_i \quad \forall\, i \in D$$

(5.11) $$\sum_{i \in D} z_i^D = v(D).$$

In their counter objection z^D, the players in $D \cap T$ claim that they can block R from carrying out their objection by inducing some players in R to join them (5.8), offering these players more than what they

were offered in the objection (5.9), while at the same time protecting their share (5.10). The players in $D \cap T$ are allowed to use the tactic of "divide and rule" by taking some members of R as partners, but they may not take all the members of R as partners (5.8).

<u>Definition 5.3</u>. A p.c. $(x,\mathcal{P})$ in a game Γ is called <u>$\mathcal{M}_c$-stable</u>, if for each coalitional objection of R against T in $(x,\mathcal{P})$, there is a coalitional counter objection of T against R.

The set $\mathcal{M}_c$ of all $\mathcal{M}_c$-stable p.c.'s in a game Γ will be called the <u>coalitional bargaining set</u> of Γ.

It is possible to strengthen the demand for stability in $\mathcal{M}_c$ and still gain something from the game.

<u>Definition 5.4</u>. A p.c. $(x,\mathcal{P})$ will be called <u>$\mathcal{M}_r$-stable</u> if for each coalitional objection of player $k \in R$ against T as in Definition 5.1, there is a coalitional counter objection by T against $k \in R$ as in Definition 5.2, except that Condition (5.8) is now replaced by

$$D \cap R \neq \emptyset, \quad k \notin D \tag{5.12}$$

Intuitively, each coalitional objection is now identified with a particular player in coalition R. The set of all $\mathcal{M}_r$-stable p.c.'s will be called the <u>restricted bargaining set</u> $\mathcal{M}_r$. Certainly $\mathcal{M}_r \subset \mathcal{M}_c$ and the following example will show that this inclusion may indeed be strict.

<u>Example 5.3</u>. Let $N = \{1,2,3\}$ and $v(i) = 0 \ \forall\ i \in N$, $v(12) = 3$, $v(23) = 5$, $v(13) = 4$ and $v(123) = 5$. An easy computation will reveal that

$$M_r = \{((2/3,\ 5/3,\ 8/3),\ (123))\} \quad \text{and}$$

$$M_c = \{((2/3,\ 5/3,\ 8/3),\ (123)),\ ((0,2,3),\ (1)(23))\}$$

The p.c. $((0,2,3),\ (1)(23))$ is not M_r-stable because a coalition objection $((\varepsilon,3-\varepsilon),\ (12))$ by player 1 against player 3 has no coalitional counter objection.

5.4 Some Examples

In this section, we present some examples which illustrate some basic differences between the bargaining sets M_r and $M_1^{(i)}$.

Example 5.4. Let $N = \{1,2,3,4\}$ and v be given by $v(i) = 0$ for all $i \in N$, $v(12) = v(13) = v(23) = 3$, $v(34) = 1$ and $v(R) = \max_{T \subset R} [v(T) + v(R-T)]$ for all other $R \subset N$. Consider p.c. $((1,1,1,1),\ (1234))$. Either player 1 or 2 or 3 can object. Let $((1+\varepsilon,2-\varepsilon),\ (12))$ be a coalitional objection to the p.c. by player 1 against coalition (34). Player 3 can counter-object by $((2,1),\ (23))$. Similarly, every coalitional objection has a coalitional counter objection. Hence $((1,1,1,1),\ (1234)) \in M_r$. A simple computation reveals that

$$M_r = \{(x,\ (1234)) : x \in \text{Conv}\{(1,1,1,1),\ (4/3,\ 4/3,\ 4/3,\ 0)\}\}.$$

Note that $M_1^{(i)}((1234)) = \{(4/3,\ 4/3,\ 4/3,\ 0)\}$ which is unreasonable as player 4 can threaten not to join the grand coalition if he is not given some share of $v(34)$ and the best that players 1, 2 and 3 can do

without player 4's cooperation is only $((1,1,1), (123))$.

Example 5.5. Consider the 6-person game given in Example 5.1. A simple computation reveals that

$$M_r = \{((2/3, 2/3, 2/3, 1, 1, 1), (123456)),$$
$$((2/3, 2/3, 2/3, 1, 1, 1), (123)(456))\}.$$

However, note that

$$M_1^{(i)}((123456)) = \mathrm{Conv}\{(2/3, 2/3, 2/3, 1, 1, 1), (1, 1, 1, 2/3, 2/3, 2/3)\}$$

which is not reasonable because

$$M_1^{(i)}((123)(456)) = \{(2/3, 2/3, 2/3, 1, 1, 1)\}$$

and the game is decomposable with partition $(123)(456)$.

Example 5.6. Consider the game $N = \{1,2,3\}$ and v given by $v(i) = 0$ for all $i \in N$, $v(12) = v(13) = 100$, $v(23) = 50$ and $v(123) = 0$. Note that the game is not superadditive. If we examine the restricted bargaining set of this game for purely theoretical reasons, we observe that $M_r = \emptyset$. A reasonable procedure, however, is to consider the superadditive cover of this game as given in Example 5.2 and the restricted bargaining set of this superadditive cover is given by

$$M_r = \{((200/3, 50/3, 50/3), (123))\}.$$

Translating this result back to the original non superadditive game (N,v), the restricted bargaining set of the game (N,v) consists of outcomes

$$\{((200/3,\ 50/3,\ 50/3),\ (12)(3)),\ ((200/3,\ 50/3,\ 50/3),\ (13)(2))\}.$$

Note that these outcomes are not payoff configurations with respect to the game (N,v). Outcome $((200/3,\ 50/3,\ 50/3),\ (12)(3))$ can be interpreted as follows: coalition (12) gives player 3 a side payment of $50/3$ in return for his cooperation in not trying to disrupt the coalition (12).

5.5 Additional Results

In this section, we present a few general results about the restricted bargaining set. Recall from Section 3.4, Chapter 3 that

$$z = \max_{P \in \Pi} w(P)$$

and

$$\Pi_z = \{P \in \Pi\colon w(P) = z\}$$

where $w(P)$ denotes the worth of c.s. P as defined in Expression (3.11). Then we obtain the following.

Theorem 5.1. Let (N,v) be an n-person superadditive game with side payments. Then $(x,P) \in M_r$ implies that $P \in \Pi_z$, i.e., the restricted bargaining set consists of only "Pareto-optimal" outcomes.

Proof: Let $(x,P) \in M_r$. Suppose $P \notin \Pi_z$. Then $\sum_{i\in N} x_i < z$. Let $\Delta = (z - \sum_{i\in N} x_i)/n$. Then $((x_1+\Delta,\ldots,x_n+\Delta),(N))$ is a coalitional

objection by any player $i \in N$ against the empty coalition $\emptyset$. But, because of Condition (5.7), there is no coalitional counter objection, which is a contradiction. $\square$

Theorem 5.2. Let Γ be a 3-person superadditive game with side payments. Then $M_r \neq \emptyset$. In fact

$$x \in M_1^{(i)}((N)) \Rightarrow (x,(N)) \in M_r.$$

Proof: Let Γ be as follows. $N = \{1,2,3\}$ and $v(12) = a$, $v(13) = b$, $v(23) = c$ and $v(123) = d$ where $0 \leq a \leq b \leq c \leq d$.

Case 1) $c \geq a+b$, $d > c$.

In this case $Co((N)) \neq \emptyset$ and the restricted bargaining set is given by

$$M_r = \{(x,(N)): x \in Co((N))\}.$$

The A-M bargaining set $M_1^{(i)}((N))$ is given by

$$M_1^{(i)}((N)) = Co((N)).$$

Case 2) $c \geq a+b$, $d = c$.

In this case, $Co((1)(23)) = Co((123)) \neq \emptyset$ and

$$M_r = \{(x,P): x \in Co((123)),\ P = (1)(23) \text{ or } (123)\}.$$

Also $M_1^{(i)}((12)(3)) = M_1^{(i)}((123)) = Co((123))$.

Case 3) $c < a+b$, $d \geq (a+b+c)/2$.

Again, $Co((123)) \neq \emptyset$ and

$$M_r = \{(x,(123)) : x \in Co((123))\}.$$

Also $M_1^{(i)}((123)) = Co((123))$.

Case 4) $c < a+b$, $d < (a+b+c)/2$.

In this case, $Co((123)) = \emptyset$ and

$$M_1^{(i)}((123)) = \{(p_1 + (d-p)/3,\ p_2 + (d-p)/3,\ p_3 + (d-p)/3)\}$$

where $p_1 = (a+b-c)/2$, $p_2 = (a+c-b)/2$, $p_3 = (b+c-a)/2$ and $p = p_1 + p_2 + p_3$. The restricted bargaining set is given by

$$M_r = \{(p_1 + (d-p)/3,\ p_2 + (d-p)/3,\ p_3 + (d-p)/3),\ (123))\}.$$

Since we have covered all cases, this completes the proof of the theorem. □

Recall the definition of $SC(S)$ given in Section 3.2. It is clear that $M_r \supset SC(Co)$. Hence if $\Pi(Co) \neq \emptyset$ for a superadditive game, $SC(Co) \neq \emptyset$ and hence $M_r \neq \emptyset$. However, at this time, the author has no general proof of existence nor a counterexample. The author conjectures that for superadditive games, $M_r \neq \emptyset$.

CHAPTER VI

SUMMARY AND CONCLUSIONS

6.1 A Summary

In the preceding pages, we have presented several theories of coalition formation. One approach was to model the process of coalition formation as an abstract game. We then studied the core and the dynamic solution of the abstract game. The predictions of the abstract game models depend on the particular payoff solution concept used. I.e., the models assume that there is a rule governing the distribution of the joint payoffs to each player in each coalition structure. The predictions of these models were then studied for the case of n-person cooperative games with side payments using various payoff solution concepts such as the individually rational payoffs, the core, the Shapley value and the bargaining set $M_1^{(i)}$. Several possible modifications of the abstract game models were also discussed.

In another approach, coalition formation was viewed as a bargaining process where the players are allowed to raise (coalitional) objections and (coalitional) counter objections in the same manner as in the Aumann-Maschler bargaining sets. While the A-M bargaining sets indicate distribution of joint payoffs given a fixed coalition structure, the restricted bargaining set indicates both formation of coalitions and distribution of payoffs as outcomes. Some examples were presented illustrating some fundamental differences in the bargaining sets $M_1^{(i)}$ and M_r. However, the important question of the existence of M_r for superadditive games is still open.

6.2 Conclusions

There are a number of interesting research problems not covered in this investigation. The abstract game models were formulated for the general case of n-person cooperative game with side payments, without side payments or in partition function form. However the predictions of the models were studied only for the special case of games with side payments. Even for this special case, the solutions of the models were characterized only for some of the known payoff solution concepts. Some of the important payoff solution concepts for which the results of the abstract game models were not considered in this work are the kernel, the nucleolus, the Banzhaf value (for simple games), the ε-core and others described in Section 1.2, Chapter 1. Other nucleoli, centers and additional value concepts, as well as several variants of the well known solution concepts mentioned also exist. It should also be interesting to investigate the predictions of these models when applied to special classes of games such as market games, quota games, convex games, symmetric games, simple games, etc.

Regarding the restricted bargaining set, an important task is to prove its existence (or to find a counterexample) for superadditive games. If a counterexample is found, the definitions of coalitional objection and coalitional counter objection may have to be modified to admit existence. A proof of existence will establish the restricted bargaining set as a viable solution concept deserving further study in regards to its mathematical structure and properties.

One more possible approach for studying the question of coalition formation in the framework of the theory of games that merits research,

is to model the process of coalition formation as a noncooperative game in normal or extensive form, in the spirit of Nash's suggestion quoted in Section 1.2, Chapter I of this work. Some recent work on noncooperative games by Harsanyi [42,43] could prove useful for this approach. Also see Lucas and Maceli [A.8].

Game theory as a mathematical tool is being increasingly employed by behavioural scientists. In the context of decision making in conflict situations of the type which can be modeled by n-person game theory, a behavioural scientist will focus on two important questions: 1) Which coalitions are likely to form? 2) How will the members of a coalition apportion their joint payoff? Although n-person game theory has largely concentrated on the second question, the behavioural scientist may well be primarily interested in the first question. Consequently, behavioural scientists have developed their own theories of coalition formation. (See Section 1.3, Chapter 1.) The main emphasis of this work has been on attempting to answer the first question in the framework of the theory of n-person cooperative games. In this respect, it is hoped that this investigation will help to make the theory of games a more attractive tool for the social scientist. The results presented in Chapter 4 reveal that some theories of coalition formation proposed by behavioural scientists are not very different from those predicted by game theory under the same assumptions, although at first glance there seems to be little resemblance between the two. Since these theories (proposed by behavioural scientists) are based on empirical observations and have been widely tested in experiments, the results of Chapter 4 may go a long way toward lessening the doubts about the relevance of game theory in predicting human behaviour.

The research presented here was motivated by the author's investigations of the world oil market. (See Shenoy [97].) In that context, an important problem was the question of the stability of the OPEC[†] cartel.

[†]Oil and Petroleum Exporting Countries, a coalition of thirteen oil producing countries.

APPENDIX

A.1 The Aumann-Dreze Generalization of the Shapley Value for all Simple Games with Four or Fewer Players

The table on the following pages contains all distinct proper simple games of four or fewer players excluding dummies. All winning coalitions are listed--the minimal winning coalitions are listed first and separated from the rest by a semicolon. The weighted voting representation given in column 4 are the simplest ones. The Shapley value Φ of a c.s. depends only on the winning coalition contained in the c.s. The Shapley value of all c.s.'s containing winning coalitions, in the sequence as in column 3, is given in column 5. The Shapley value of a c.s. not containing any winning coalition is zero for each player and therefore is not given in column 5. Column 6 contains all c.s.'s in $K_0(\Phi)$ identified by the winning coalition it contains. The last column indicates whether the game exhibits the paradox of smaller coalitions or not.

A.2 n-Person Games in Partition Function Form

Let $N = \{1,\ldots,n\}$ be a set of n players who are represented by $1,\ldots,n$. Let $P = (P_1,\ldots,P_m)$ be an arbitrary partition of N into coalitions $P_1,\ldots,P_m$. Then for each partition P, assume there is an outcome function

$$F : P \to E^1$$

which assigns the real numbered outcome $F_P(P_j)$ to the coalition P_j

Number of Players	$\mathcal{W}$	Weighted Voting Representation	Φ	$K_0(\Phi)$	Paradox?
1	A	[1; 1]	(1)	A	no
2	AB	[2; 1,1]	(1/2, 1/2)	AB	no
3	ABC	[3; 1,1,1]	(1/3, 1/3, 1/3)	ABC	no
3	AB,AC;ABC	[3; 1,1,1]	(1/2, 1/2, 0), (1/2, 0, 1/2), (2/3, 1/6, 1/6)	AB,AC,ABC	yes
3	AB,AC,BC; ABC.	[2; 1,1,1]	(1/2, 1/2, 0), (1/2, 0, 1/2), (0, 1/2, 1/2), (1/3, 1/3, 1/3)	AB,AC,BC	no
4	ABCD	[4; 1,1,1,1]	(1/4, 1/4, 1/4, 1/4)	ABCD	no
4	ABC,ABD; ABCD	[5; 2,2,1,1]	(1/3, 1/3, 1/3, 0), (1/3, 1/3, 0, 1/3), (5/12, 5/12, 1/12, 1/12)	ABC,ABD, ABCD	yes
4	ABC,ABD ACD;ABCD	[4; 2,1,1,1]	(1/3, 1/3, 1/3, 0), (1/3, 1/3, 0, 1/3), (1/3, 0, 1/3, 1/3), (1/2, 1/6, 1/6, 1/6)	ABC,ABD, ACD,ABCD	yes
4	ABC,ABD, ACD,BCD; ABCD	[3; 1,1,1,1]	(1/3, 1/3, 1/3, 0), (1/3, 1/3, 0, 1/3), (1/3, 0, 1/3, 1/3), (0, 1/3, 1/3, 1/3), (1/4, 1/4, 1/4, 1/4)	ABC,ABD, ACD,BCD	no
4	AB,ACD; ABD;ABC, ABCD	[5; 3,2,1,1]	(1/2, 1/2, 0, 0), (1/3, 0, 1/3, 1/3), (1/2, 1/2, 0, 0), (1/2, 1/2, 0, 0), (7/12, 3/12, 1/12, 1/12)	AB,ABD, ABC,ABCD	yes
4	AB,ACD, BCD;ABD ABC,ABCD	[4; 2,2,1,1]	(1/2, 1/2, 0, 0), (1/3, 0, 1/3, 1/3), (1, 1/3, 1/3, 1/3), (1/2, 1/2, 0, 0), (1/2, 1/2, 0, 0), (1/3, 1/3, 1/6, 1/6)	AB,ABD,ABC	no

Table A.1. The Aumann-Dreze generalization of the Shapley value for simple games.

Number of Players	$\mathcal{W}$	Weighted Majority Representation	Φ	$K_0(\Phi)$	Paradox?
4	AB,AC, BCD;ACD, ABD,ABC, ABCD	[5; 3,2,2,1]	(1/2, 1/2, 0, 0), (1/2, 0, 1/2, 0), (0, 1/3, 1/3, 1/3), (1/2, 0, 1/2, 0), (1/2, 1/2, 0, 0), (2/3, 1/6, 1/6, 0), (5/12, 3/12, 3/12, 1/12)	AB,ABD,ABC	yes
4	AB,AC,AD; ABC,ABD, ACD,ABCD	[4; 3,1,1,1]	(1/2, 1/2, 0, 0), (1/2, 0, 1/2, 0), (1/2, 0, 0, 1/2), (2/3, 1/6, 1/6, 0), (2/3, 1/6, 0, 1/6), (2/3, 0, 1/6, 1/6), (9/12, 1/12, 1/12, 1/12)	ABC,ABD,ACD, ABCD	yes
4	AB,AC,AD, BCD;ABC, ABD,ACD, ABCD	[3; 2,1,1,1]	(1/2, 1/2, 0, 0), (1/2, 0, 1/2, 0), (1/2, 0, 0, 1/2), (0, 1/3, 1/3, 1/3), (2/3, 1/6, 1/6, 0), (2/3, 1/6, 0, 1/6), (2/3, 0, 1/6, 1/6), (1/2, 1/6, 1/6, 1/6)	$\emptyset$	yes

Table A.1. (continued)

when the partition P forms. The function

$$F: \Pi \to F_P$$

which assigns to each partition its outcome function is called the partition function for the game. Finally, the ordered pair

$$\Gamma = (N,F)$$

is called an n-person game in partition function form.

A.3 Additional References

A.1. J. F. Banzhaf, "Multi-member electoral districts - Do they violate the "one man, one vote" principle," Yale Law J., 75, 1966, pp. 1309-1338.

A.2. J. F. Banzhaf, "One man, 3.312 votes: A mathematical analysis of the electoral college," Villanova Law Review, 13, 1968, pp. 303-346.

A.3. J. F. Banzhaf, "One man, ? votes: Mathematical analysis of political consequences and judicial choices," The George Washington Law Review, 36, 1968, pp. 808-823.

A.4. J. S. Coleman, "Control of collectivities and the power of a collectivity to act," Social Choice, B. Lieberman, ed., Gordon and Breach, N.Y., 1971, pp. 269-300.

A.5. P. Dubey, "Probabilistic generalizations of the Shapley value," Cowles Foundation Discussion Paper No. 440, Yale University, New Haven, Conn., 1976.

A.6. P. Dubey and L. S. Shapley, "Mathematical properties of the Banzhaf power index," The Rand Corporation, Santa Monica, Calif., to appear.

A.7. S. Groennings, E. W. Kelley and M. Leiserson, eds., The Study of Coalition Behaviour, Holt, Rinehart and Winston, N.Y., 1970.

A.8. W. F. Lucas and J. C. Maceli, "Discrete partition function games," Technical Report No. 344, School of Operations Research and Industrial Engineering, Cornell University, Ithaca, N.Y., 1977.

A.9. R. D. Mckelvey, P. C. Ordeshook and M. D. Winer, "The competitive solution for n-person games without transferable utility, with an application to committee games," Amer. Pol. Science Review, to appear.

A.10. R. D. Mckelvey and P. C. Ordeshook, "Competitive coalition theory," Prepared for the M.S.S.B. conference on Game Theory and Political Science, Hyannis, Mass., July, 1977.

A.11. G. Owen, "Multilinear extensions and the Banzhaf value, Nav. Res. Logist. Quart., 18, 1971, pp. 345-355.

A.12. G. Owen, "Characterization of the Banzhaf-Coleman index," Prepared for the M.S.S.B. Conference on Game Theory and Political Science, Hyannis, Mass., July, 1977.

A.13. A. E. Roth, "Bargaining ability, the utility of playing a game, and models of coalition formation," J. Math. Psychology, to appear.

A.14. L. S. Shapley, "Simple games: An outline of the descriptive theory," Behavioural Science, 7, 1962, pp. 59-66.

A.15. L. S. Shapley, "A comparison of power indices and a nonsymmetric generalization," Rand Paper P-5872, The Rand Corporation, Santa Santa Monica, Calif., 1977.

A.16. E. Vilkas, "Optimality and dynamics in coalitional games," Mathematical Economics and Game Theory, Essays in Honor of Oskar Morgenstern, ed. by R. Henn and O. Moeschlin, Lecture Notes in Economics and Mathematical Systems, Vol. 141, pp. 181-198, Springer Verlag, N.Y., 1977.

BIBLIOGRAPHY

1. S. Agmon, "The relaxation method for linear inequalities," Canadian J. Math., 6, 1954, pp. 382-392.

2. S. L. Anderson and E. A. Traynor, "An application of the Aumann-Maschler n-person cooperative game," Recent Advances in Game Theory, Proceedings of a conference held in Princeton in October 1961, privately printed for members of the conference, M. Maschler, ed., Princeton Univ. Conf., Princeton, N.J., 1962, pp. 265-270.

3. N. Asscher, "An ordinal bargaining set for games without side payments," Math. of Operations Research, 1, 1976, pp. 381-389.

4. N. Asscher, "A cardinal bargaining set for cooperative games without side payments," Int. J. Game Theory, to appear.

5. R. J. Aumann, "Linearity of unrestrictedly transferable utilities," Nav. Res. Logist. Quart., 7, 1960, pp. 281-284.

6. R. J. Aumann, "The core of a cooperative game without side payments," Trans. Amer. Math. Soc., 98, 1961, pp. 539-552.

7. R. J. Aumann, "A survey of cooperative games without side payments," Essays in Mathematical Economics in Honor of Oskar Morgenstern, M. Shubik ed., Princeton Univ. Press, Princeton, N.J., 1967, pp. 3-27.

8. R. J. Aumann and J. H. Dreze, "Cooperative games with coalition structures," Int. J. Game Theory, 3, 1974, pp. 217-237

9. R. J. Aumann and M. Maschler, "The bargaining set for cooperative games," Advances in Game Theory, M. Dresher, L. S. Shapley, and A. W. Tucker, eds., Annals of Mathematics Studies No. 52, Princeton University Press, Princeton, N.J., 1964, pp. 443-476.

10. R. J. Aumann and B. Peleg, "von Neumann-Morgenstern solutions to cooperative games without side payments," Bull. Amer. Math. Soc., 66, 1960, pp. 173-179.

11. R. Axelrod, Conflict of Interest, A Theory of Divergent Goals with Applications to Politics, Markham Publishing Co., Chicago, Ill., 1970.

12. J. F. Banzhaf, "Weighted voting doesn't work: A mathematical analysis," Rutgers Law Review, 19, 1965, pp. 317-343.

13. M. Behzad and F. Harary, "On the problem of characterizing digraphs with solutions and kernels," Bull. Iranian Math. Soc., 3, 1975.

14. M. Behzad and F. Harary, "Which directed graphs have a solution?," Mathematica Slovaca, 27, 1977, pp. 37-41.

15. L. J. Billera, "Existence of general bargaining sets for cooperative games without side payments," Bull. Amer. Math. Soc., 76, 1970, pp. 375-379.

16. L. J. Billera, "Some theorems on the core of an n-person game without side payments," SIAM J. Appl. Math., 18, 1970, pp. 567-579.

17. L. J. Billera, "Some recent results in n-person game theory," Math. Prog., 1, 1971, pp. 58-71.

18. L. J. Billera, "Global stability in n-person games," Trans. Amer. Math. Soc., 172, 1972, pp. 45-46.

19. O. N. Bondareva, "Theory of the core in the n-person game," Leningrad State University, Vestnik L.G.U., 13, 1962, pp. 141-142. (In Russian.)

20. O. N. Bondareva, "Some applications of linear programming methods to the theory of cooperative games," Problemy Kibernetiki, 10, 1963, pp. 119-139. (In Russian.)

21. S. J. Brams, Paradox in Politics, Free Press, N.Y., 1976.

22. T. Caplow, "A theory of coalitions in the triad," American Sociological Review, 21, 1956, pp. 489-493.

23. T. Caplow, "Further development of a theory of coalitions in the triad," American J. Sociology, 64, 1959, pp. 488-493.

24. T. Caplow, Two Against One: Coalitions in Triads, Prentice-Hall, Englewood Cliffs, N.J., 1968.

25. J. M. Chertkoff, "A revision of Caplow's coalition theory," J. Experimental Social Psychology, 3, 1967, pp. 172-177.

26. J. G. Cross, "Some theoretic characterisations of economic and political coalitions," J. Conflict Resolution, 11, 1967, pp. 184-195.

27. M. Davis, "A bargaining procedure leading to the Shapley value," Econometric Research Memorandum 61, Princeton Univ., Princeton, N.J., 1961.

28. M. Davis and M. Maschler, "The kernel of a cooperative game," Nav. Res. Logist. Quart., 12, 1965, pp. 223-259.

29. M. Davis and M. Maschler, "Existence of stable payoff configurations for cooperative games," Essays in Mathematical Economics in Honor of Oskar Morgenstern. M. Shubik ed., Princeton Univ. Press, Princeton, N.J., 1967, pp. 39-52.

30. P. Dubey, "Some results on values of finite and infinite games," Technical Report, Center for Applied Mathematics, Cornell University, Ithaca, N.Y. 14853, 1975.

31. R. L. Eisenman, "Alliance games of n-persons," Nav. Res. Logist. Quart., 13, 1966, pp. 403-411.

32. R. L. Eisenman, "On solutions of alliance games," Ph.D. Dissertation, University of Michigan, Ann Arbor, Mich., 1964.

33. D. Fink, "On a solution concept for multiperson cooperative games," Int. J. Game Theory, to appear.

34. W. A. Gamson, "A theory of coalition formation," American Sociological Review, 26, 1961, pp. 373-382.

35. D. B. Gillies, "Solutions to general non-zero-sum games," Contributions to the Theory of Games, Vol. IV, A. W. Tucker and R. D. Luce, eds., Annals of Math. Studies No. 40, Princeton Univ. Press, Princeton, N.J., 1959, pp. 47-85.

36. J. H. Grotte, "Dynamics of cooperative games," Int. J. Game Theory, 5, 1976, pp. 27-64.

37. F. Harary, Graph Theory, Addison-Wesley, Reading, Mass., 1969.

38. J. C. Harsanyi, "Approaches to the bargaining problem before and after the theory of games: a critical discussion of Zeuthen's, Hicks' and Nash's theories," Econometrica, 24, 1956, pp. 144-157.

39. J. C. Harsanyi, "A bargaining model for the cooperative n-person game," Contributions to the Theory of Games, Vol. IV, A. W. Tucker and R. D. Luce, eds., Annals of Math. Studies No. 40, Princeton Univ. Press, Princeton, N.J., 1959, pp. 325-355.

40. J. C. Harsanyi, "A simplified bargaining model for the n-person cooperative game," Int. Econ. Review, 4, 1963, pp. 194-220.

41. J. C. Harsanyi, "An equilibrium-point interpretation of stable sets and a proposed alternative definition," Management Science, 20, 1974, pp. 1472-1495.

42. J. C. Harsanyi, "The tracing procedure: a Bayesian approach to defining a solution for n-person noncooperative games," Int. J. Game Theory, 4, 1975, pp. 61-94.

43. J. C. Harsanyi, "A solution concept for n-person noncooperative games," Int. J. Game Theory, 6, 1977, to appear.

44. J. C. Harsanyi and R. Selten, "A generalized Nash Solution for the two-person bargaining games with incomplete information," Management Science, 18, 1972, pp. 80-106.

45. J. R. Isbell, "A modification of Harsanyi's bargaining model," Bull. Amer. Math. Soc., 66, 1960, pp. 70-73.

46. E. Kalai, E. A. Pazner and D. Schmeidler, "Collective choice correspondences as admissible outcomes of social bargaining processes," Econometrica, 44, 1976, pp. 223-240.

47. E. Kalai and D. Schmeidler, "An admissible set occurring in various bargaining situations," Discussion Paper No. 191, The Center for Mathematical Studies in Economics and Management Science, Northwestern University, Evanston, Ill., 1975.

48. G. Kalai, M. Maschler and G. Owen, "Asymptotic stability and other properties of trajectories and transfer sequences leading to the bargaining sets," Int. J. Game Theory, 4, 1975, pp. 193-213.

49. S. S. Komorita and J. M. Chertkoff, "A bargaining theory of coalition formation," Psychological Review, 80, 1973, pp. 149-162.

50. S. S. Komorita, "A weighted probability model of coalition formation," Psychological Review, 81, 1974, pp. 242-256.

51. M. Leiserson, "Coalitions in politics," Ph.D. Dissertation, Yale University, New Haven, Conn., 1966.

52. W. F. Lucas, "Solutions for four-person games in partition function form," SIAM J. Appl. Math., 13, 1965, pp. 118-128.

53. W. F. Lucas, "Solution theory for n-person games in partition function form," Game Theory: Techniques and Applications, Proceedings of a NATO Conference (Toulon, 1964), A. Mensch, ed., English University Press, London, 1966, pp. 131-134.

54. W. F. Lucas, "Solutions for a class of n-person games in partition function form," Nav. Res. Logist. Quart., 14, 1967, pp. 15-21.

55. W. F. Lucas, "A counterexample in game theory," Management Science, 13, 1967, pp. 766-767.

56. W. F. Lucas, "A game in partition function form with no solution," SIAM J. Appl. Math., 16, 1968, pp. 582-585.

57. W. F. Lucas, "On solutions for n-person games," Research Memorandum, RM-5567-PR, The Rand Corp., Santa Monica, Calif., 1968.

58. W. F. Lucas, "A game with no solution," Bull. Amer. Math. Soc., 74, 1968, pp. 237-239.

59. W. F. Lucas, "The proof that a game may not have a solution," Trans. Amer. Math. Soc., 137, 1969, pp. 219-229.

60. W. F. Lucas, "Games with unique solutions that are nonconvex," Pacific J. Math., 28, 1969, pp. 599-602.

61. W. F. Lucas, "Some recent developments in n-person game theory," SIAM Review, 13, 1971, pp. 491-523.

62. W. F. Lucas, "An overview of the mathematical theory of games," Management Science, 18, 1972, pp. 3-19.

63. W. F. Lucas, "Disconnected solutions," Bull. Amer. Math. Soc., 82, 1976, pp. 596-598.

64. W. F. Lucas, "Measuring power in weighted voting systems," Case Studies in Applied Mathematics, ed. by M. D. Thompson, The Mathematical Association of America, Washington, D.C., Chapter 3, pp. 42-106, 1976.

65. W. F. Lucas, "The existence problem for solutions," Mathematical Economics and Game Theory, Essays in Honor of Oskar Morgenstern, ed. by R. Henn and O. Moeschlin, Lecture Notes in Economics and Mathematical Systems, Vol. 141, pp. 64-75, Springer-Verlag, N.Y., 1977.

66. R. D. Luce and H. Raiffa, Games and Decisions: Introduction and Critical Survey, John Wiley and Sons, Inc., N.Y., 1957.

67. M. Maschler and B. Peleg, "Stable sets and stable points of set-valued dynamic systems with applications to game theory," SIAM J. Control and Optimization, 14, 1976, pp. 985-995.

68. J. W. Milnor, "Reasonable outcomes for n-person games," Research Memorandum RM-916, The Rand Corp., Santa Monica, Calif., 1952.

69. K. Miyasawa, "The n-person bargaining game," Advances in Game Theory, M. Dresher, L. S. Shapley and A. W. Tucker, eds., Annals of Math. Studies No. 52, Princeton Univ. Press, Princeton, N.J., 1964.

70. R. B. Myerson, "Graphs and cooperation in games," Math. of Operations Research, to appear.

71. J. F. Nash, "Equilibrium points in n-person games," Proc. Nat. Acad. Science, USA, 36, 1950, pp. 48-49.

72. J. F. Nash, "The bargaining problem," Econometrica, 18, 1950, pp. 155-162.

73. J. F. Nash, "Noncooperative games," Annals of Math., 54, 1951, pp. 286-295.

74. J. F. Nash, "Two person cooperative games," Econometrica, 21, 1953, pp. 128-140.

75. J. von Neumann and O. Morgenstern, Theory of Games and Economic Behaviour, Princeton Univ. Press, Princeton, N.J., 1944; 2nd ed., 1947; 3rd ed., 1953.

76. G. Owen, "Multilinear extensions of games," Management Science, 18, 1972, pp. 64-79.

77. G. Owen, "Values of games without side payments," Int. J. Game Theory, 1, 1972, pp. 95-109.

78. G. Owen, "A note on the convergence of transfer sequences in n-person games," Int. J. Game Theory, 4, 1975, pp. 221-228.

79. B. Peleg, "Solutions to cooperative games without side payments," Trans. Amer. Math. Soc., 106, 1963, pp. 280-292.

80. B. Peleg, "Bargaining sets of cooperative games without side payments," Israel J. Math., 1, 1963, pp. 197-200.

81. B. Peleg, "Existence theorem for the bargaining set," Essays in Mathematical Economics in Honor of Oskar Morgenstern, M. Shubik, ed., Princeton Univ. Press, N.J., 1967, pp. 53-56.

82. A. Rapoport, N-Person Game Theory, The Univ. of Michigan Press, Ann Arbor, Mich., 1970.

83. W. H. Riker, The Theory of Political Coalitions, Yale Univ. Press, New Haven, Conn., 1962.

84. A. E. Roth, "Subsolutions and the supercore of cooperative games," Math. of Operations Research, 1, 1976, pp. 43-49.

85. A. E. Roth, "Utility functions for simple games," Faculty Working Paper #305, College of Commerce and Business Administration, University of Illinois, Urbana-Champaign, Ill., 1976.

86. H. Scarf, "The core of an n-person game," Econometrica, 35, 1967, pp. 60-69.

87. D. Schmeidler, "The nucleolus of a characteristic function game," SIAM J. Appl. Math., 17, 1969, pp. 1163-1170.

88. R. Selten, "A simple model of imperfect competition, where 4 are few and 6 are many," Int. J. Game Theory, 2, 1973, pp. 141-201.

89. L. S. Shapley, "A value for n-person games," Contributions to the Theory of Games, Vol. II, H. W. Kuhn and A. W. Tucker, eds., Annals of Math. Studies No. 28, Princeton Univ. Press, Princeton, N.J., 1953, pp. 307-317.

90. L. S. Shapley, "A solution containing an arbitrary closed component," Contributions to the Theory of Games, Vol. IV, A. W. Tucker and R. D. Luce, eds., Annals of Math. Studies No. 40, Princeton Univ. Press, Princeton, N.J., 1959, pp. 47-85.

91. L. S. Shapley, "Notes on n-person games. VIII: a game with infinitely 'flaky' solutions," Unpublished manuscript.

92. L. S. Shapley, "Values of large market games: status of the problem," Research Memorandum RM3957PR, The Rand Corp., Santa Monica, Calif., 1964.

93. L. S. Shapley, "On balanced sets and cores," Nav. Res. Logist. Quart., 14, 1967, pp. 453-460.

94. L. S. Shapley and M. Shubik, "A method for evaluating the distribution of power in a committee system," Amer. Pol. Science Review, 48, 1954, pp. 787-792.

95. P. Shenoy, "A dynamic solution concept for cooperative games," Unpublished manuscript, 1976.

96. P. Shenoy, "Coalition structure solutions of cooperative games," Unpublished manuscript, 1976.

97. P. Shenoy, "Game-theoretical models of the world oil market," Technical Report, School of Operations Research and Industrial Engineering, Cornell University, Ithaca, N.Y., 1977.

98. K. Shmadich, "The existence of graphic solutions," The Leningrad University Herald, 1, 1976, pp. 88-92. (In Russian.)

99. M. Shubik, Strategy and Market Structure: Competition, Oligopoly, and the Theory of Games, John Wiley & Sons, Inc., N.Y., 1959.

100. R. Stearns, "Three-person cooperative games without side payments," Advances in Game Theory, M. Dresher, L. S. Shapley, and A. W. Tucker, eds., Annals of Math. Studies No. 52, Princeton Univ. Press, Princeton, N.J., 1964.

101. R. E. Stearns, "A game without side payments that has no solution," Report of the Fifth Conference on Game Theory, W. F. Lucas, ed., Dept. of Mathematics, mimeographed, Princeton, N.J., 1965, out of print.

102. R. E. Stearns, "Convergent transfer schemes for n-person games," Trans. Amer. Math. Soc., 134, 1968, pp. 449-459.

103. P. D. Straffin, "Power indices in politics," Modules in Applied Mathematics, The Math. Association of America, 334 Upson Hall, Cornell University, Ithaca, N.Y., 14853, 1976.

104. P. D. Straffin, "Probability models for measuring voting power," Technical Report No. 320, School of Operations Research and Industrial Engineering, Cornell University, Ithaca, N.Y., 1976.

105. M. Taylor, "Review article: Mathematical political theory," British J. Pol. Science, 1, 1971, pp. 339-382.

106. R. M. Thrall, "Generalized characteristic functions for n-person games," Proceedings of the Princeton University Conference of October, 1961, privately printed for members of the conference, 1962, pp. 157-160.

107. R. M. Thrall and W. F. Lucas, "n-person games in partition function form," Nav. Res. Logist. Quart., 10, 1963, pp. 281-298.

108. W. Vickrey, "Self-policing properties of certain imputation sets," Contributions to the Theory of Games, Vol. IV, A. W. Tucker and R. D. Luce, eds., Annals of Math. Studies No. 40, Princeton Univ. Press, Princeton, N.J., 1959.

109. W. E. Vinacke and A. Arkoff, "An experimental study of coalitions in the triad," American Sociological Review, 22, 1957, pp. 406-414.

110. L. S.-Y. Wu, "A dynamic theory for the class of games with nonempty cores," SIAM J. Appl. Math., 32, 1977, pp. 328-338.

111. L. S.-Y. Wu and L. J. Billera, "On a dynamic theory for the kernel of an n-person game," Int. J. Game Theory, to appear.

112. R. J. Weber, "Bargaining solutions and stationary sets in n-person games," Technical Report No. 223, Dept. of Operations Research, Cornell University, Ithaca, N.Y., 1974.